# MECHANICS

MODULAR SYSTEM

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MODULAR SYSTEM

# MECHANICS

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# PREFACE



*Dear teachers,*

**Mechanics** has been written specially for Zambak Modular System. There is one book corresponding to each subject of high school Physics in this Modular System. This book has been prepared for high school and college students to allow flexibility and variety in use.

The subjects and their order have been carefully organized by an international committee to allow for wide access in different educational systems. However, teachers are strongly encouraged to follow the order and methods of the content presentation since they have been carefully written according to the best pedagogical approach.

There are sufficient examples, questions and problems in each chapter in order to avoid the use of other resources for homework assignments. This book also has a valuable supplementary resource material, **'Mechanics Question Bank'**, consisting of multiple choice questions to enhance practice in the course.

The language in the text has been edited to be accessible by students whose second language is English. Problems are related to everyday applications, to arouse interest in the topics.

Visual aids; full colour diagrams, tables and photographs are used wherever possible to make the content more intuitively understandable via the link with everyday experience.

Quantities are given according to the metric system in SI units. The approximate value of g (the acceleration of free-fall or the gravitational field strength) is taken to be 10 $m/s^2$. Answers are usually given to the correct number of significant figures, but this is sometimes ignored in the answers of the problems sections, for simplicity.

*Dear students,*
The key to obtaining the most benefit from this book is as follows: Read the **Introduction** text in the beginning of each chapter to get an idea of what will be studied. Recite each physical law or an important term that appears in **this type style**. Work on **Examples** and check your answers with **Solutions** to put your fresh knowledge into practice. Use the **Summary** section as a revision checklist. Note that conceptual **Questions and Problems** are in a specific order according to the main stream of the chapter and the level of difficulty. You can find the **Answers** to numerical problems at the end of the chapter. Supplementary materials such as mathematical theories, physical constants and important formulae are provided in the **Appendixes**. A trigonometric table is provided in case you don't have a scientific calculator on hand. To search for a specific term, use the **Index** at the end of the book.

The authors thank all members of Zambak Publications, in particular Ramazan ŞAHİN, B.Yüksel ŞAHAN, Metin SÜLÜ, Osman ÖZPALA, Murat BAYCAN, Musa BAŞ and R. İNCE for their help in producing this book. Great thanks and appreciation to Edip TÜRK for his self-sacrificing work in the design and type setting of the pages. Finally thanks to our families for their patience and understanding during the preparation period of this book.

***The authors***

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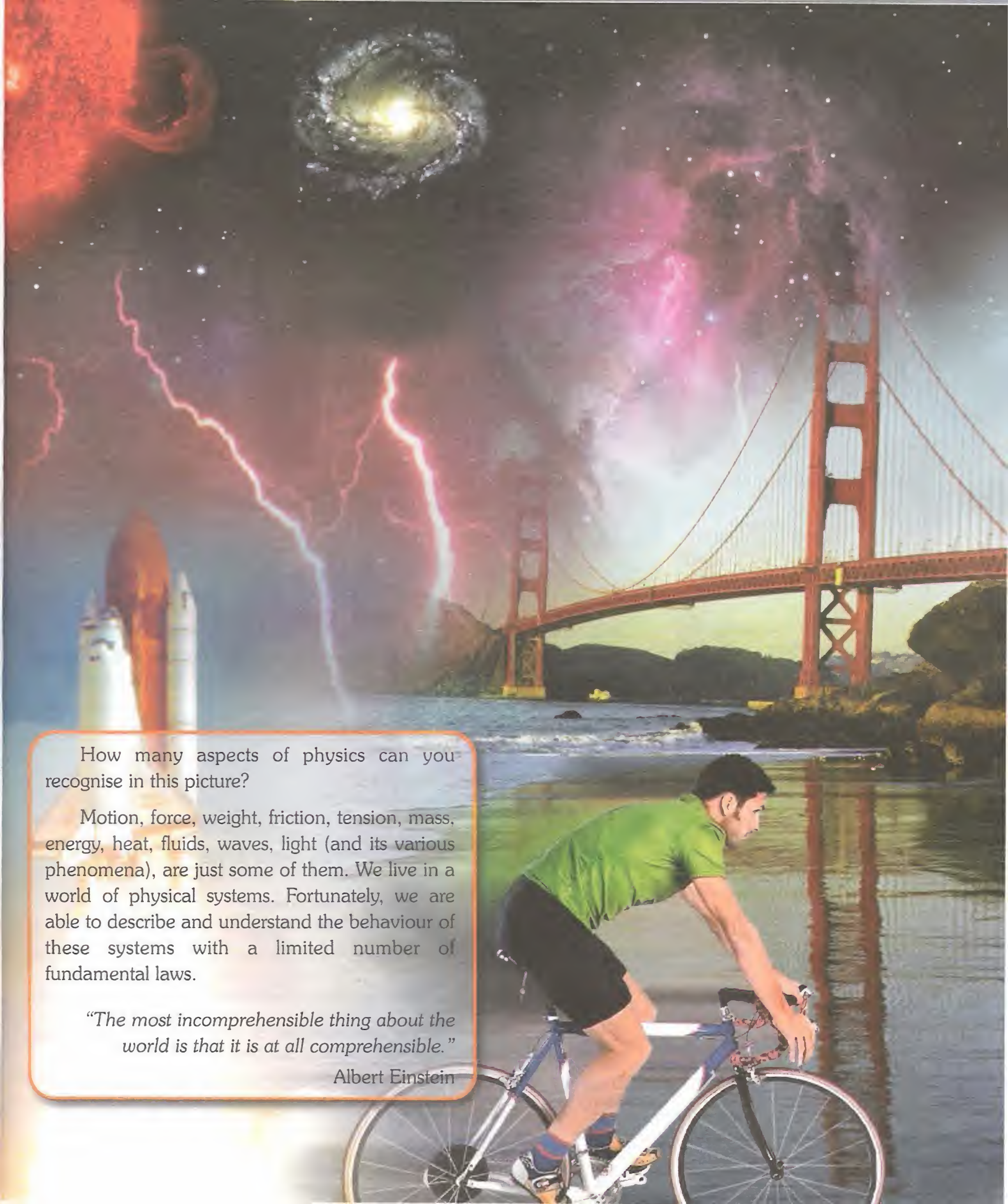


How many aspects of physics can you recognise in this picture?

Motion, force, weight, friction, tension, mass, energy, heat, fluids, waves, light (and its various phenomena), are just some of them. We live in a world of physical systems. Fortunately, we are able to describe and understand the behaviour of these systems with a limited number of fundamental laws.

*"The most incomprehensible thing about the world is that it is at all comprehensible."*

Albert Einstein

CHAPTER 1

# Units and Physical Quantities


*Prior to studying the following chapters, let's start with a clear definition of the basic quantities involved in the physical laws.*

## 1.1 STANDARDS AND UNITS

Measurement is an important activity not only in physics but also in our everyday life. Let's say you would like to prepare yourself an omelette. You find the recipe as follows:

*Mix some corn starch with a lot of eggs, add a little milk and some butter, bake for a while...*

How helpful do you think this is?

Now consider this one:

*Mix 5 eggs with 30 ml of milk, add 20 g. butter and bake for 3 minutes.*

Now you know the quantities of each ingredient. So measurements are clear and objective if they are described with numbers. But is this always enough?

Imagine that you are told to measure the length of your classroom. Rather than saying very long or too short, you take a measurement of it using the size of your

feet as a unit. Let's say you walk the length of the classroom with one foot in front of the other, which takes 32 footsteps. This measurement is not consistent because someone with larger sized feet will measure the same length to be, say, 28 times the length of their feet. Measurements must be the same for everyone. That is why physical quantities are expressed using **standard units**.
In conclusion, each measured physical quantity must have two parts; a number (numerical value) and a standard unit.

**Physical quantity**

**7.5** **m**

numerical value — standard unit

## a. The SI System of Units

The group of internationally accepted standard units is called **the metric system** or 'the international system', abbreviated as **SI** from 'Le Systeme International d'Unites' in French. In this system, each measurement is a comparison with the fundamental SI unit of the related quantity. By expressing the length of an object to be, say, 7.5 metres you are actually making a comparison. Everybody acknowledges the length of the room to be 7.5 times as long as the fundamental unit of length, defined as 1.0 metre.

### Length, Time and Mass

Force, velocity, and acceleration are some examples of physical quantities. The units of these quantities and all others are combinations of the units of the base quantities. Table 1.1 shows the 7 base quantities and their units. The first three are commonly used in Mechanics. These are;

**kilogram** **(kg)** for Mass,
**metre** **(m)** for Length
**second** **(s)** for Time.

| Base Quantity | Base Unit | |
|---|---|---|
| | Name | Symbol |
| mass | kilogram | kg |
| length | metre | m |
| time | second | s |
| temperature | kelvin | K |
| current | ampere | A |
| amount of substance | mole | mol |
| luminous intensity | candela | cd |

**Table 1.1** Base quantities in physics and their units.

## b. Derived Units

The units of other physical quantities can be derived from those listed in Table 1.1. For example, let's try to obtain the SI unit of density which is defined by the following equation:

$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

Mass is a base quantity measured in kg, but volume is not. Since volume is defined by **length × width × height**, it has another derived unit of $m^3$. Therefore, using the defining equation, the derived unit of density is $kg/m^3$.

Scientists usually give special names to the complicated combinations. For example the unit of force is the **newton (N)** which stands for the derived unit of $kg{\cdot}m/s^2$.

## Example 1.1 — Derived unit of pressure

Find out the derived unit for pressure, using the derived unit of force, $kg{\cdot}m/s^2$, and the following equation, $\text{pressure} = \dfrac{\text{force}}{\text{area}}$

### Solution

Substituting the units into the equation gives the derived unit of pressure: $\dfrac{kg \cdot m/s^2}{m^2} = kg/ms^2$ which is given a special name: the pascal (Pa)

## Example 1.2 — Unit of gravitational field strength

The period (T) of a simple pendulum is defined as the time taken to complete one full cycle. It is given by the following equation:

$T = 2\pi\sqrt{\frac{\ell}{g}}$ where $\ell$ is the length, and g is the gravitational field strength. Derive the SI unit of g.

**Solution**

Using the defined equation;

$$T^2 = 4\pi^2\frac{\ell}{g}$$

$$g = \frac{4\pi^2\ell}{T^2}$$

The constant "$4\pi^2$" has no unit.

$\ell$ (length) has a base unit of m;

T (period) has a derived unit of s;

So $T^2$ has a unit of $s^2$;

Therefore the derived unit of g is $\frac{m}{s^2}$

| Quantity | Dimension | SI unit |
|---|---|---|
| Area | $L^2$ | $m^2$ |
| Volume | $L^3$ | $m^3$ |
| Speed | $L/T$ | m/s |
| Acceleration | $L/T^2$ | $m/s^2$ |
| Force | $M\cdot L/T^2$ | kg $m/s^2$ |
| Energy | $M\cdot L^2/T^2$ | $kg\cdot m^2/s^2$ |

**Table 1.2** Dimensions of some physical quantities and their SI units.

### c. Dimensional Analysis

Both sides of an equation must have the same units, that is the equation must be **homogeneous.** Using this fact, a technique called **dimensional analysis** can be applied when we need to check a specific formula or an equation.

In physics, each physical quantity represents its nature via its dimension. For example, the distance between two points, d, has the dimension of Length (L) regardless of its unit in any system. This is shown as [d]=L. Thus, physical quantities can be expressed via these basic dimensions: for example the dimensions of volume can be written as $[V]=L^3$.

The other two dimensions commonly use d in mechanics are Mass (M), and Time (T). Table 1.2 lists some physical quantities along with their dimensions and SI units.

## Example 1.3 — Dimensional analysis of force

Show that force has the dimensions of $[F]=\frac{ML}{T^2}$

**Solution**

Force is defined by the equation;

Force=mass x acceleration

Acceleration is not a base quantity so it is

defined by $\text{acceleration} = \frac{\text{change in velocity}}{\text{time}}$

So $\text{force} = \text{mass} \times \frac{\text{change in velocit}}{\text{time}}$

Using $\text{velocity} = \frac{\text{displacement}}{\text{time}}$,

$$\text{force} = \text{mass} \times \frac{\text{displacement}}{(\text{time})^2}$$

the dimensions of velocity are $[v]=\frac{L}{T}$ therefore,

Therefore $[F]= M\frac{[v]}{T} = M\frac{L}{T\cdot T} = \frac{ML}{T^2}$

## Example 1.4 Validating an equation

Show that the equation $x = \frac{1}{2}at^2$ is dimensionally correct where (x) is the distance moved by an object (initially at rest) in time (t) at a constant acceleration of (a).

**Solution**

The dimensional form of this equation is

$$[x] = \frac{1}{2}[a][t]^2$$

$$L = \frac{L}{T^2}T^2$$

$$L = L$$

Both sides of the equation have the same dimensions. Therefore the equation is valid.

Note that the constant $\frac{1}{2}$ in the equation is a dimensionless quantity and our analysis works for any constant in the equation. Dimensional analysis gives no information about the magnitude of the constants.

## d. Significant Figures

There is always an uncertainty in a measurement. The number used to describe the physical quantity also implies how precisely it has been measured via the number of **significant figures**. This is the number of separate digits expressed when stating it. For example 9.3 has two separate digits, a nine and a three, thus, it is said to have two significant figures. Assume that an object is measured to be 9.3 cm using a ruler accurate to the nearest mm (0.1 cm). The actual length lies somewhere between 9.2 cm and 9.4 cm. We can express this as 9.3 cm ± 0.1 cm where ± 0.1 cm is the uncertainty in the measurement. Here, the measured value of 9.3 cm has two significant figures. Using the same ruler, it is not possible to express the length as, say, 9.32 cm. Since 9.32 cm has three significant figures and this implies that a precision of 1/100th of a centimetre (0.01 cm) can be measured.

| 1 s.f. | 2 s.f. | 3 s.f. |
|---|---|---|
| 0.8 kg | 8.4 kg | 8.40 kg |
| $3\times10^{-2}$ cm | $3.0\times10^{-2}$ cm | 0.0302 cm |
| 7 s | 7.0 s | 7.03 s |

Table 1.3 Numbers expressed as one, two and three significant figures.

Numbers are usually expressed in **standard form** to clearly denote the number of significant figures. To find the number of significant figures, the total number of digits can be counted from left to right, starting from the first non-zero digit. See some examples in Table 1.3.

Remember that when multiplying or dividing numbers, the result is always expressed in terms of the lowest number of significant figures expressed for any of the inputs. In addition and subtraction, the answer is expressed according to the lowest number of decimal places expressed for any of the inputs.

## Example 1.5 Using the correct number of significant figures

The density of iron is 7800 kg/m$^3$. Using the equation for density, calculate the volume of an iron block of mass 5.2 kg.

**Solution**

Density is defined by the equation; $\text{density} = \frac{\text{mass}}{\text{volume}}$

Therefore; $\text{volume} = \frac{\text{mass}}{\text{density}} \Rightarrow \text{volume} = \frac{5.2\,\text{kg}}{7800\,\text{kg/m}^3}$

$\text{volume} = 0.0006667\ \text{m}^3$

This is the number that will be displayed on a calculator. However, how many significant figures should be used?

The density of iron was expressed as 4 significant figures, but its mass was expressed as 2 significant figures.

Thus, the answer cannot be precise to more than 2 significant figures, the answer must be stated in standard form as follows;

volume = $6.7 \times 10^{-4}$ m$^3$ (2 s.f.)



Figure 1.1 A honeybee flaps its wings almost 250 times in one second. In other words, it flaps its wings once every 4 *milli*seconds.

## e. Prefixes

Standard prefixes are used for very large or very small quantities. The most common ones are shown in the table below:

| $10^{-12}$ | $10^{-9}$ | $10^{-6}$ | $10^{-3}$ | $10^{3}$ | $10^{6}$ | $10^{9}$ |
|---|---|---|---|---|---|---|
| pico (p) | nano (n) | micro (μ) | milli (m) | kilo (k) | mega (M) | giga (G) |

For example, the frequency of an FM radio band, 102.8 x $10^6$ Hertz, is abbreviated as 102.8 MHz (mega-hertz). A honey bee flaps its wings once every 0.004 s or $4\times10^{-3}$ s, which can be abbreviated as 4 ms (milliseconds.)

# 1.2 VECTORS

Science is based on experiments and measurements. Therefore the results of these measurements should be stated in a definite and clear way. For example, let's say that the air temperature is 24 °C. This is satisfying because it expresses a numerical value with a proper unit. However, let's say that a plane has travelled a distance of 100 km and then landed. This is an incomplete expression because the direction of travel is not given. It is unclear at what city it has landed.

Hence, physical quantities are divided into two main groups. One group of quantities has only magnitude (a numerical value) along with their unit. These are called **scalars**. The second group of quantities are described by both magnitude and direction. These are called **vectors**.

Table 1.4 gives examples of some common scalar and vector quantities.

| scalars | vectors |
|---|---|
| mass | weight |
| speed | velocity |
| distance | displacement |
| energy | force |
| temperature | acceleration |
| charge | electric field |

Table 1.4 Some examples of scalar and vector quantities.

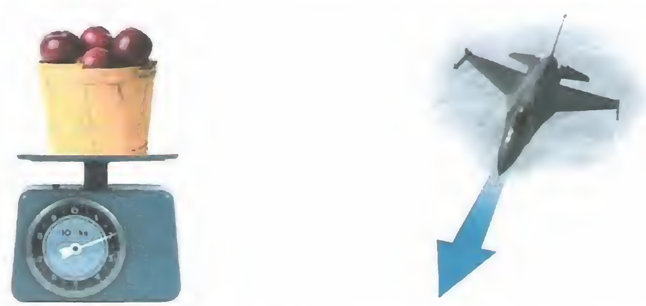


Figure 1.2 The mass of a 2 kg apple is scalar and the velocity of the plane 900 km/h to the west, is a vector.

## a. Representing Vectors

Vectors are drawn as arrows. The direction of the arrow, not surprisingly, shows the direction of the vector. The length of the arrow is usually drawn proportional in size to the size (magnitude) of the vector. Velocity vectors of various magnitudes are sketched in Figure 1.3. In this book, vectors are expressed with an arrow over the top of their symbols.

The magnitude of a vector $\vec{A}$ is denoted either by A or $|A|$. By definition, the magnitude of a vector is always positive.

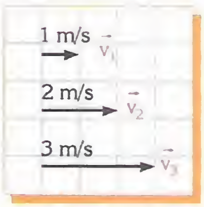



Figure 1.3 Vectors are represented by arrows, the magnitude of the vector is proportional to the size of the arrow.

## b. Equality of Vectors

Two vectors are equal if they have the same magnitude and direction. That is also to say that a vector can be moved parallel to itself without changing its size and direction. Vectors $\vec{A}$ and $\vec{B}$ in Figure 1.4 are equal.

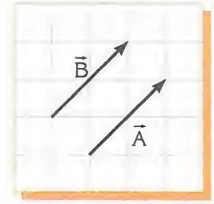



**Figure 1.4** Two equal vectors.

## c. Adding Vectors

Physical quantities, both scalars and vectors, should have the same units in order to be added. In addition to this, vectors are added and subtracted differently than scalars since they have a direction as well as a magnitude.

For instance, mass is a scalar quantity. If you add 3 kg of apple to 4 kg of apple, the total mass is always 7 kg. However, two force vectors, 3 N and 4 N don't always add up to 7 N. See Figure 1.5.

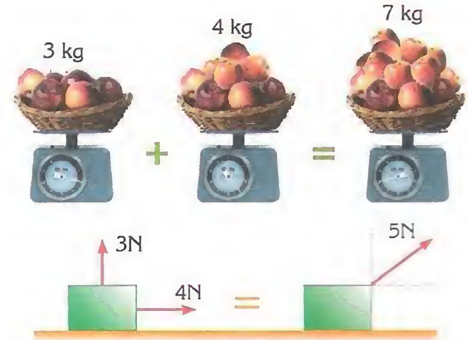



**Figure 1.5** Vector addition is different to scalar addition

There are different methods to add and subtract vectors:

### 1) Triangle Method

When two vectors, $\vec{A}$ and $\vec{B}$ are added together, the tail of the second vector B, is placed at the head of the first vector. The sum is called the **resultant** or **effective vector**, $\vec{R}$, and is drawn from the free tail (the tail of vector $\vec{A}$) to the free head (the head of vector $\vec{B}$). This is shown as


See also how to add two parallel vectors in Figure 1.6 (a) and (b)

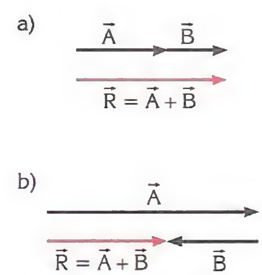



**Figure 1.6** Adding parallel vectors a) when they point in the same direction b) when they point in opposite directions.

### 2) Polygon Method

The method used above can also be applied to add more than two vectors. Draw all vectors one by one with the head of each at the tail of the next one, and complete the polygon by joining the tail of the first vector to the head of the last one. This will give you the resultant vector. See Figure 1.7. Note that the result doesn't change when the order of construction changes.

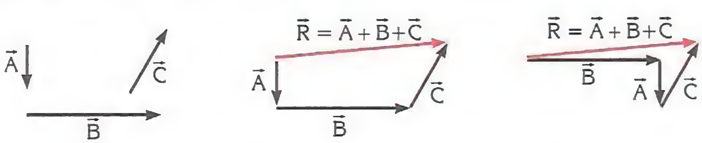



**Figure 1.7** Adding vectors using the polygon method is independent of the order of construction.

### 3) Parallelogram Method

Two vectors can also be added by an alternative method: Vectors are re-drawn with their tails joined to each other, and a parallelogram is obtained by drawing their parallel sides. The resultant vector is the diagonal of this parallelogram. See Figure 1.8.

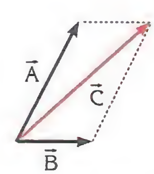
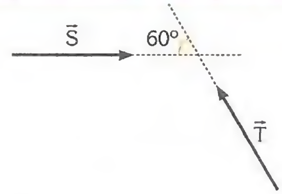



**Figure 1.8** Adding vectors using the parallelogram method

## Example 1.6 Adding two vectors using a diagram

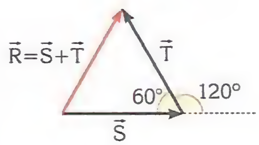
Vectors $\vec{S}$ and $\vec{T}$ have the same magnitude, as shown in the figure. Find the resultant vector $\vec{R} = \vec{S}+\vec{T}$ using a diagram.




**Solution**

a) Triangle Method

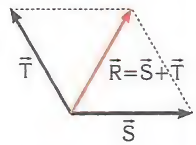
Vectors $\vec{S}$ and $\vec{T}$ are re-drawn, as shown in the figure. Vector $\vec{R}$ is drawn from the tail of vector $\vec{S}$ to the head of vector $\vec{T}$




b) Parallelogram Method

Vectors are drawn by joining their tails. Parallel sides are drawn in. The diagonal of this parallelogram is the resultant vector.

Note that both methods result in the same answer.



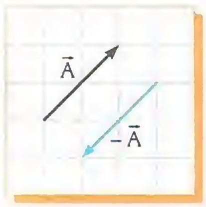




**Figure 1.9** Negative of a vector

### d. The Negative of a Vector

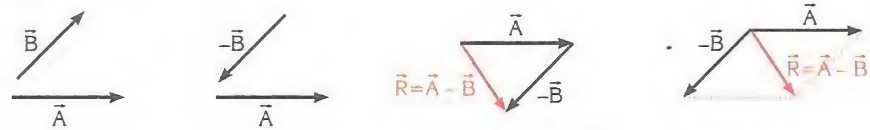
A vector and its negative have the same magnitude but opposite directions, as shown in Figure 1.9. To obtain the negative of a vector, re-draw the original with the same size but in the opposite direction.

### e. Subtraction of Vectors

Vector subtraction is a special case of vector addition. Let us consider two vectors $\vec{A}$ and $\vec{B}$. The vector difference $\vec{A}-\vec{B}$ is the same as adding vector $\vec{A}$ to the negative of vector $\vec{B}$. That is,

$$\vec{A}-\vec{B} = \vec{A}+(-\vec{B})$$

The triangle or parallelogram method can be applied to find the resultant $\vec{R}$. (Figure 1.10)

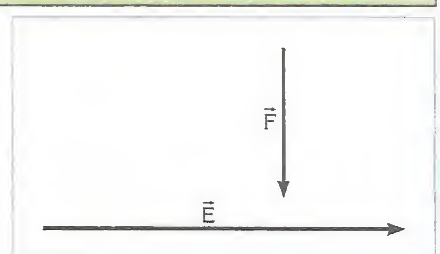



$\vec{A}-\vec{B}=?$ | First, draw the negative of B | Then use either the triangle method | or the parallelogram method

**Figure 1.10** Subtraction of vectors

## Example 1.7 Subtraction of perpendicular vectors

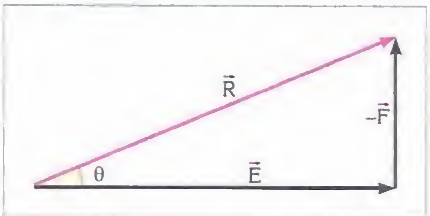
Vectors $\vec{E}$ and $\vec{F}$ of magnitudes 12 units and 5 units, respectively, are shown in the figure. Find the magnitude and direction of the resultant vector $\vec{R} = \vec{E} - \vec{F}$.




**Solution**

Since $\vec{E} - \vec{F} = \vec{E} + (-\vec{F})$, the negative of vector $\vec{F}$ can be found.

If the triangle method is applied by adding vector $\vec{E}$ with vector $-\vec{F}$ the following diagram is obtained




Using the Pythagorean theorem;

$$R^2 = E^2 + F^2 = 12^2 + 5^2$$

$$R^2 = 169 \Rightarrow R = 13 \text{ units}$$

From $\tan\theta = \dfrac{F}{E} \Rightarrow \tan\theta = \dfrac{5}{12} \Rightarrow \tan\theta = 0.42$

$\theta = 23°$.

As a result, $\vec{R}$ has a magnitude of 13 units at an angle of 23° with the horizontal.

## f. Multiplication and Division of Vectors by Scalars

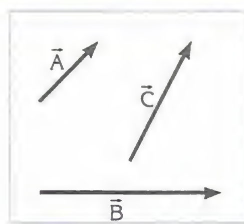
Vector quantities can be multiplied by scalars. The resultant is also a vector. Assume a vector $\vec{A}$ is multiplied by a scalar k. The direction of the resultant vector alters, depending on the sign of k. If k is positive, the direction of vector $\vec{A}$ doesn't change, but if k is negative, the resultant vector will be in the opposite direction to vector $\vec{A}$. See some examples in Figure 1.11.


Figure 1.11 Multiplication and division of vector $\vec{A}$ by scalars.

## Example 1.8

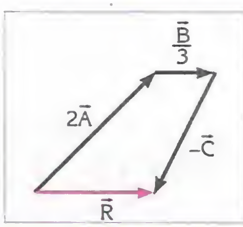
Multiplication of vectors by scalars

Vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are shown in the figure. Draw the resultant vector $\vec{R} = 2\vec{A} + \dfrac{\vec{B}}{3} - \vec{C}$.




**Solution**

To obtain the solution, the vectors $2\vec{A}$, $\dfrac{\vec{B}}{3}$ and $-\vec{C}$ can be added using the polygon method.




Therefore the resultant vector $\vec{R}$ is in the +x direction with a magnitude of 4 units.

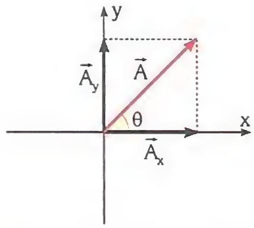



**Figure 1.12** Vector $\vec{A}$ is resolved into its two perpendicular components $\vec{A}_x$ and $\vec{A}_y$.

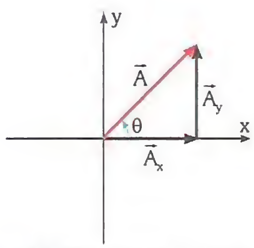



**Figure 1.13** The vector triangle shows that $\vec{A}$ is the addition of $\vec{A}_x$ and $\vec{A}_y$. Using the trigonometric relationship in this right angled triangle, the components can be calculated.

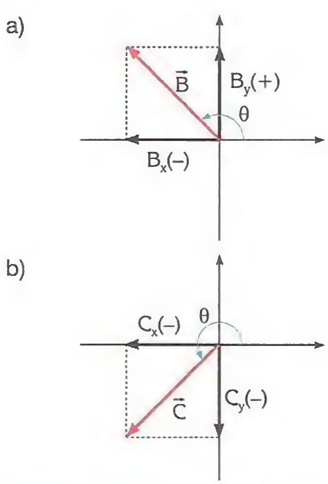



**Figure 1.14** a) and b) The magnitude of a vector is always positive. However, its components may have negative values since the angle is measured anti-clockwise from the +x-axis.

## g. Components of a Vector

So far vectors have been added using a scale drawing or using the properties of right angled triangles. Scale drawing has some limitations, and calculations using right angled triangles (the use of the Pythagorean theorem) is only applicable to perpendicular vectors. This method would be too complicated for more than two vectors with an angle other than 90°. Therefore a simple but more general method is required for the addition (and subtraction) of vectors. An alternative method is the use of components.

Generally speaking, each vector can be said to be the sum of two perpendicular vectors. These vectors, called **components**, are the projections of the original vector along the x and y axes of a rectangular coordinate system. In other words, each vector can be resolved into its x and y components, which are perpendicular to each other.

Examine vector $\vec{A}$ in Figure 1.12. To find its components, the tail of vector $\vec{A}$ is first placed at the origin of an x-y coordinate system. The projections along the horizontal (x-axis) and vertical (y-axis) are obtained by drawing horizontal and vertical lines from the tip of the vector to the x and y axes.

The projection onto the x-axis is called the x component and denoted by $\vec{A}_x$, whereas the projection onto the y-axis is called the y component and denoted by $\vec{A}_y$.

The vector sum of these components is equal to $\vec{A}$.

$$\vec{A} = \vec{A}_x + \vec{A}_y$$

See Figure 1.13. Applying the Pythagorean Theorem to this right angled triangle, the relationship between the magnitudes of A, $A_x$ and $A_y$ can be written as;

$$A^2 = A_x^2 + A_y^2$$

Remember we said that the magnitude of a vector is always defined to be positive. The components of a vector, on the other hand, may be positive or negative numbers. If the x component of vector $\vec{A}$ points in the +x-direction, $A_x$ is defined to be the magnitude of $\vec{A}_x$. If $\vec{A}_x$ points in the -x direction, $A_x$ is defined to be the negative of that magnitude. Examine Figure 1.14.a which describes a vector with a negative x-component. Figure 1.14.b shows a vector with both of its components negative.

Trigonometry can be used to calculate the components.

Examine Figure 1.13 again. If the angle between vector $\vec{A}$ and vector $\vec{A}_x$ is θ;

$$\cos\theta = \frac{A_X}{A} \qquad \text{and} \qquad \sin\theta = \frac{A_Y}{A}$$

Hence, $A_x = A\cos\theta$ $\quad A_y = A\sin\theta$ *

* Note that these equations are valid when the angle θ is measured anti-clockwise from the +x axis. For instance, the cosine of the angle θ in Figure 1.14.a is negative, therefore the component $B_x$ of this vector, which is along the -x axis, is also negative.

Let's now look at how to use components to find the resultant of several vectors. In Figure 1.15 the vector sum $\vec{R}$ of two vectors $\vec{A}$ and $\vec{B}$ are found by using the components of each vector along the x and y directions. It's clear from the figure that the x-component of $\vec{R}$, $\vec{R}_x$, is the addition of $\vec{A}_x+\vec{B}_x$, and similarly the y-component $\vec{R}_y$ is the addition of $\vec{A}_y+\vec{B}_y$. That is;

$$R_x = A_x + B_x \qquad R_y = A_y + B_y$$

The same procedure can be applied to add any number of vectors.

If $\vec{R}$ is the resultant of $\vec{A}+\vec{B}+\vec{C}+\vec{D}+.....$ Then,

$$R_x = A_x+B_x+C_x+D_x+.....$$
$$R_y = A_y+B_y+C_y+D_y+....$$

Note that using the components, all problems involving vector addition can be reduced to the addition of $\vec{R}_x$ and $\vec{R}_y$.

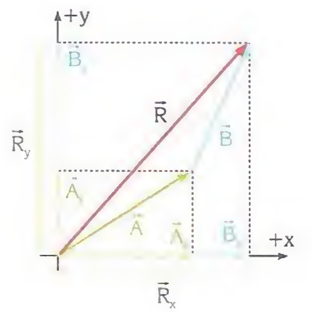
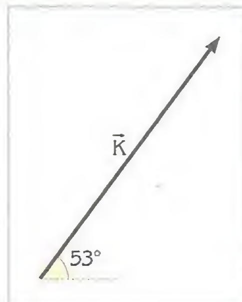



**Figure 1.15** $\vec{R}$ is the vector sum of $\vec{A}$ and $\vec{B}$. The x-component of $\vec{R}$ is the sum of the x-components of $\vec{A}$ and $\vec{B}$. Similarly the y-component of $\vec{R}$ is the sum of the y-components of $\vec{A}$ and $\vec{B}$.

## Example 1.9 — Components of a vector

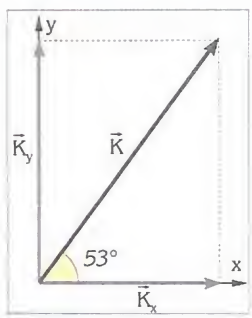
Find the magnitude of the components $\vec{K}_x$ and $\vec{K}_y$ of vector $\vec{K}$ which has a magnitude of 10 units and makes an angle of 53° with the horizontal, as shown in the figure. (cos 53°=0.6; sin 53°=0.8)




### Solution

Using the angle that vector $\vec{K}$ makes with the x axis, the magnitudes of components $K_x$ and $K_y$ can be found;

$$K_x = K \cos 53° \qquad K_y = K \sin 53°$$
$$K_x = 10 \,.\, 0.6 \qquad K_y = 10 \,.\, 0.8$$
$$K_x = 6 \text{ units} \qquad K_y = 8 \text{ units}$$




Note that according to the Pythagorean theorem

$$K^2 = K_x^2 + K_y^2$$

## Problem Solving Strategy

In general, two or more vectors can be added with no difficulty using the following procedure which includes the Pythagorean theorem, and simple trigonometric functions:

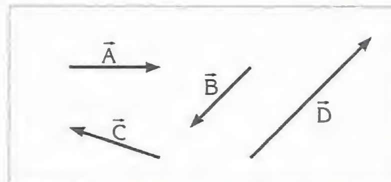
- Each vector is placed on the coordinate system.
- The x and y components of all vectors are found.
- The resultant components are calculated. (Making a table for the algebraic sum of the components is useful. See Example 1.10)
- The Pythagorean theorem is used to find the magnitude of the resultant vector.
- The angle that the resultant vector makes with an axis is found using a suitable trigonometric function.

This strategy is usually sufficient to calculate answers to most problems. There are also alternative methods for adding vectors. For example, Cosine and Sine Rules are some other powerful mathematical tools used in vector addition. See Appendix A to review these rules and study Example 1.11.

## Example 1.10 — Addition of vectors using the component method

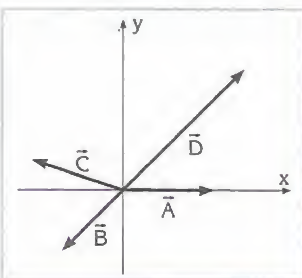
Vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ are shown in the figure. Find the magnitude and direction of the resultant vector

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$$




**Solution**

Place the vectors on the same coordinate system to find their components.




| Vector | x-component | y-component |
|---|---|---|
| A | 3 | 0 |
| B | –2 | –2 |
| C | –3 | 1 |
| D | 4 | 4 |
| R | 2 | 3 |

$$R_x = A_x + B_x + C_x + D_x$$

$$R_x = 3 - 2 - 3 + 4 = 2 \text{ units}$$

$$R_y = A_y + B_y + C_y + D_y$$

$$R_y = 0 - 2 + 1 + 4 = 3 \text{ units}$$

The magnitude of vector $\vec{R}$ is found from the equation

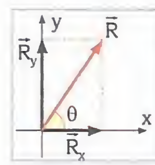
$$R^2 = R_x^2 + R_y^2$$

$$R^2 = 2^2 + 3^2$$

$$R^2 = 3.6 \quad \text{units.}$$

And its direction is given by,

$$\tan\theta = \frac{R_y}{R_x} = \frac{3}{2}$$




From the trigonometric table

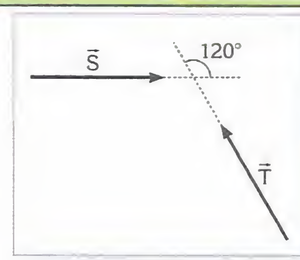
$$\theta = 56.3°$$

So the resultant vector $\vec{R}$, has a magnitude of 3.6 units, and makes an angle of 56.3° with the +x axis.

## Example 1.11 — Addition of intersecting vectors

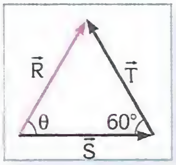
Vectors $\vec{S}$ and $\vec{T}$ having the same magnitude of 8 units are drawn in the figure.

Find the magnitude and direction of the resultant vector $\vec{R} = \vec{S} + \vec{T}$.




**Solution**

Let us redraw the vectors $\vec{T}$ and $\vec{S}$ by joining the head of $\vec{S}$ to the tail of $\vec{T}$.




The magnitude of vector $\vec{R}$, that is drawn from the free tail to the free head, can be found using the cosine rule;

$$R^2 = S^2 + T^2 - 2ST\cos 60°$$

$$R^2 = 8^2 + 8^2 - 2 \cdot 8 \cdot 8 \cdot 0.5$$

$$R = 8 \text{ units}$$

If the angle that vector $\vec{R}$ makes with the +x axis is denoted as θ, from the sine rule;

$$\frac{T}{\sin\theta} = \frac{R}{\sin 60°}$$

$$\frac{8}{\sin\theta} = \frac{8}{\sin 60°}$$

$$\theta = 60°$$

Thus, the resultant vector $\vec{R}$ has a magnitude of 8 units and makes an angle of 60° with the +x axis.

# Summary

Each measured physical quantity consists of two parts; a number (numerical value) and a standard unit. The group of internationally accepted standard units is called the metric system, abbreviated as SI.

In this system,

kilogram **(kg)** for mass;

metre **(m)** for length;

second **(s)** for time;

are 3 base units that are used to derive other units in mechanics.

Each physical quantity is represented by its dimension. A technique called dimensional analysis can be used to check whether an equation is in the correct form.

The number of significant figures implies how precisely a quantity has been measured.

Standard prefixes are used for very large or very small numbers.

Vectors are physical quantities described by both magnitude and also direction. They can be represented by arrows.

There are various methods to add or subtract two vectors.

They can be placed head-to-tail triangle and Polygon methods or they can be joined to each other from their tails and a parallelogram constructed from them. The diagonal of this parallelogram is the resultant vector.

The negative of a vector is drawn with the same size but in the opposite direction to the vector.

Vector subtraction is a special case of vector addition.

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

Vectors can be multiplied by scalars. The resultant vector depends on the magnitude and the sign of that scalar.

A vector can be resolved into its rectangular components by finding their projections on the x and y axes. Using the Pythagorean theorem,

$$A^2 = A_x^2 + A_y^2$$

From the triangle formed by the vector and its components,

$$A_x = A\cos\theta \quad \text{and} \quad A_y = A\sin\theta$$

where θ is the angle between $\vec{A}$ and $\vec{A}_x$.

# QUESTIONS AND PROBLEMS

## 1.1 Standards and Units

1. The kinetic Energy KE of a mass, m, moving at a speed of v is given by the following equation:

$$KE = \frac{1}{2}mv^2$$

What is the unit of energy expressed in base units? What special name is given to this combination of base units?

2. Power is defined as the time rate of energy change. (power= energy/time) Determine the derived SI unit of power. What special name is given to this?

3. The gravitational force, F between two objects of mass $m_1$ and $m_2$ is defined by $F = G\frac{m_1 m_2}{r^2}$ where r is the distance measured between the centres of the objects and G is known as the gravitational constant.

What are the base SI units of G?

4. Elastic potential energy stored in a spring extended by x metres is given by $E = \frac{1}{2}kx^2$ where k is the spring constant. Find the derived SI unit of k.

5. In November 2005, astronomers discovered a massive star. Its mass was eight times larger than that of the Sun and the star was only 30 million years old. Calculations showed that it is moving extremely quickly, at a speed of 2.6 million km per hour. Calculate its speed in metres per second (m/s).

6. Using Table 1.2, show that Einstein's famous equation of energy (E) and mass (m), $E=mc^2$, where c is the speed of light, is dimensionally correct.

7. The square of the speed of an object moving at a constant acceleration (a) in time (t) over a displacement (x) is given by $v^2=ka^mx^n$ where k, m, and n are dimensionless constants. Use dimensional analysis to find m and n.

8. You are given a spring balance having divisions of 1 mg. Which of the following may be a measurement read directly from this spring balance?

0.7 kg    0.68 kg    0.675 kg    0.6748 kg

9. Scientists believe that the universe was formed from a tiny point by a huge explosion called the Big Bang. Stephen Hawking, the famous physicist, points out that "If the rate of expansion one second after the Big Bang had been smaller by even *one part in a hundred thousand million million*, the universe would have recollapsed before it ever reached its present size" Express that fractional number mentioned above in standard form using a suitable prefix.

10. Determine the number of significant figures (s.f.) in the following numbers: +

a) 161 kg

b) 11.339 s

c) $3.0 \times 10^8$ m/s

d) 0.0052 m

11. Everything around us is made up of atoms. An atom is a million times smaller than the thickness of a human hair. The diameter of an atom ranges from about 0.1 to 0.5 nanometres (nm). Express this range in millimetres (mm).

12. Measure how many times your heart beats in each minute. Estimate the number of heart-beats during an average life span of 60 years.

13. Plan an investigation to estimate the thickness of one piece of paper in this book.

+14. When asked about how long his lecture takes, a physics teacher replies as follows: "Don't worry, it won't take more than a micro century!". How long does he mean by this in minutes?

15. Rewrite the following quantities using a suitable prefix:

a) 0.003 s

b) 8900 g

c) $6.02 \times 10^9$ m

d) 18 000 000 J

e) 0.00005 m

1.2 Vectors

16. Identify whether each of the following examples involves a vector or a scalar quantity.

a) The force applied by one boxer on another

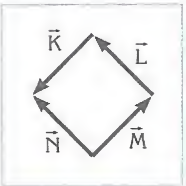
b) The number of pages in a book

c) The volume of some milk

d) The distance that a ball travels

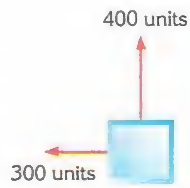
e) The velocity of a bullet

17. Which pair of vectors $\vec{K}$, $\vec{L}$, $\vec{M}$ and $\vec{N}$, shown in the figure are equal to each other?

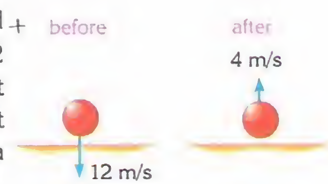


**18.** Vector $\vec{P}$ has a magnitude of 10 units. Vector $\vec{Q}$ has a magnitude of 8 units. Find the largest and smallest values possible for the resultant vector $\vec{R}=\vec{P}+\vec{Q}$

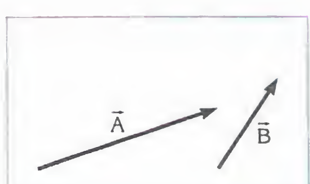
**19.** A heavy box is pulled by two ropes. One rope pulls it with a force of 300 units due west. The other pulls it with a force of 400 units due north. Draw a vector diagram and calculate the magnitude and direction of the resultant force.



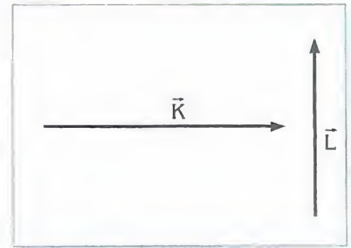

**20.** A ball falls freely and reaches a speed of 12 m/s just before it strikes the ground. It rebounds vertically at a speed of 8 m/s. Calculate the change in its velocity.




**21.** Two vectors are given in the figure. Copy these vectors into your notebook and find the resultant $\vec{R}=\vec{A}+\vec{B}$ using a scale drawing.

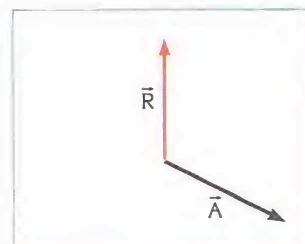



**22.**



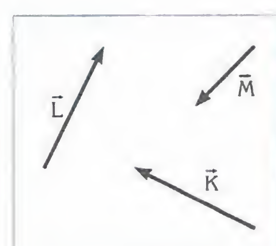

Vectors $\vec{K}$ and $\vec{L}$ of magnitudes 8 units and 6 units, respectively, are as shown in the figure. Calculate the magnitude and direction of the resultant vector $\vec{R}=\vec{K}+\vec{L}$.

**23.**




$\vec{R}$ is the resultant of two vectors, $\vec{A}$ and $\vec{B}$. ($\vec{R}=\vec{A}+\vec{B}$). The vectors $\vec{R}$ and $\vec{A}$ are shown in the figure, find the vector $\vec{B}$.

**24.**

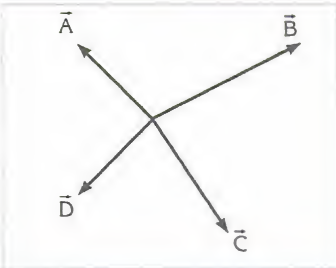



Vectors $\vec{K}$, $\vec{L}$ and $\vec{M}$ are as shown in the figure. Draw the resultant vectors given below.

a) $\vec{R} = \vec{K} + \vec{L} - \vec{M}$

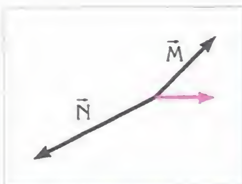
b) $\vec{R} = \vec{K} - \vec{L} - \vec{M}$

**25.**




Draw the resultant of vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ placed as shown in the figure.

**26.** The resultant of vectors $\vec{M}$, $\vec{N}$ and $\vec{P}$ is $\vec{R}$. The vectors $\vec{M}$, $\vec{N}$ and $\vec{R}$ are shown in the figure, find vector $\vec{P}$.

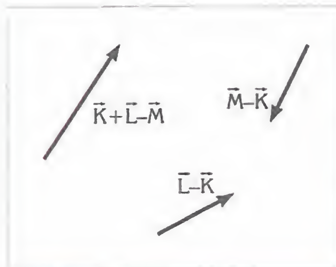



**27.** Under what condition can the sum of three vectors of equal magnitude be zero?

**28.** Two forces $\vec{F}_1$ and $\vec{F}_2$ act at the origin of the xy plane. $\vec{F}_1$ has a magnitude of 25 units in the direction of the positive y axis. The second force $\vec{F}_2$ has a magnitude of 50 units in a direction 30° below the negative x axis. Find the resultant vector $\vec{R}=\vec{F}_1+\vec{F}_2$ using a scale drawing.

**29.** Vectors $2\vec{A}$ and $-\vec{B}$ are as shown in the figure. Draw the resultant vector $\vec{R}=\vec{A}+\vec{B}$.


**30.**

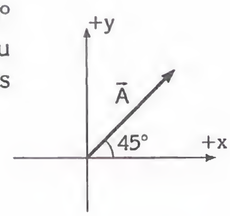



Vectors $\vec{K}+\vec{L}-\vec{M}$, $\vec{M}-\vec{K}$ and $\vec{L}-\vec{K}$ are as shown in the figure. Find vector $\vec{K}$.

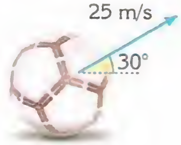
## g. Components of a vector

**31.** In which region of the x-y coordinate system will the components of a vector have opposite signs?

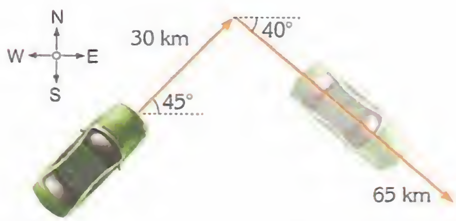
**32.** If a vector makes an angle of 45° with the +x axis, what can you say about the magnitudes of its components?



33. A ball is kicked at a speed of 25 m/s at an angle of 30° to the ground. What are the horizontal and vertical components of its initial velocity?




34.




A car travels 30 km northeast. Then, on the second day it travels 65 km in a direction 40° south of east. Calculate the components of its displacement for each day.

35. A honey bee flies 70° north of west from its hive for 100 m. How far would it have to fly due east and due south to arrive back at its hive?

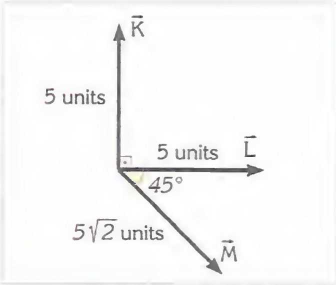
36. A force vector of $\vec{A}$ has a positive x component of magnitude 5.0 units and a negative y component of magnitude 8.0 units.

a) Calculate the magnitude and direction of $\vec{A}$.

b) When another force vector $\vec{B}$ is added to vector $\vec{A}$ it gives a resultant vector of magnitude 2.0 units in the positive x direction. Find vector $\vec{B}$.

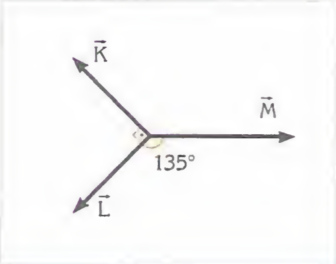
37. $\vec{A}$ and $\vec{B}$ are two perpendicular vectors. What can you say about the component of $\vec{B}$ along the direction of vector $\vec{A}$?

38.



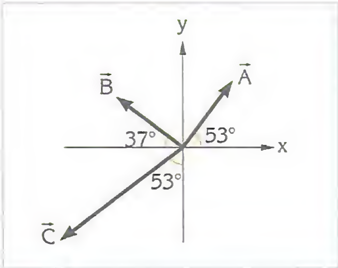

Vectors $\vec{K}$, $\vec{L}$ and $\vec{M}$ are shown in the figure. Find the magnitude and direction of their resultant vector.

39.




The resultant of vectors $\vec{K}$, $\vec{L}$ and $\vec{M}$ is zero. The magnitudes of vectors $\vec{K}$ and $\vec{L}$ are both 5 units. Find the magnitude of the vector $\vec{M}$.

40.




The magnitudes of vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ shown in the figure are 10 m, 10 m and 45 m, respectively. Find the magnitude and direction of the vector $\vec{A}+3\vec{B}-\frac{1}{2}\vec{C}$.

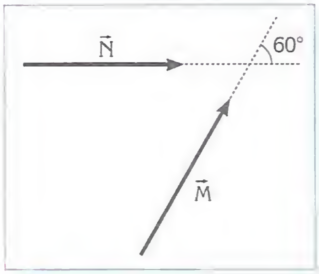
41. Vectors $\vec{A}$ and $\vec{B}$ have x and y components as follows:

$A_x=+3$ units $\quad B_x=-5$ units

$A_y=+6$ units. $\quad B_y=+4$ units

If $\vec{B}-\vec{A}+2\vec{C}=0$, what are the components of vector $\vec{C}$?

42.

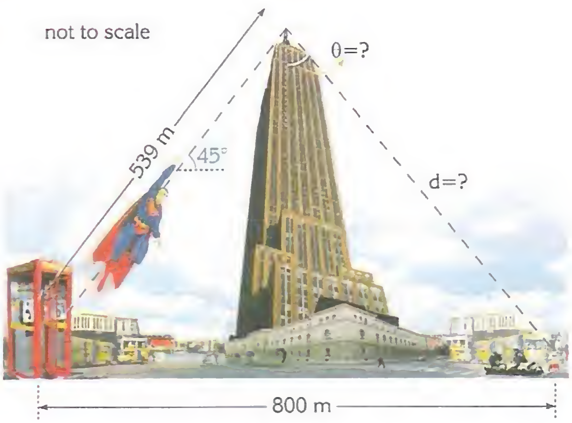



Vectors $\vec{M}$ and $\vec{N}$, having the same magnitude of 5 units, are given in the figure. Calculate the magnitude and direction of the resultant vector $\vec{R}=\vec{M}+\vec{N}$.

43.


A plane flies to Constanta 500 km due east from the airport. Then it flies from Constanta to Bucharest 225 km northwest. Draw a vector diagram and determine the straight-line distance and direction from the airport to Bucharest.

44.




Superman flies from a phone booth to the top of the Empire State Building, a total distance of 539 m in a direction of 45° to the horizontal.

a) Calculate the height of the Empire State Building.

b) At the top of the building he sees someone in danger 800 m away from, and on the same level as, the phone booth. How far (d=?) and in what direction (θ=?) should superman fly downwards to save him?

CHAPTER 2

# Motion in One Dimension

*We live in a perpetually moving universe. Movement occurs at all scales, from the sub-microscopic world of electrons in atoms, to the astronomical world of the galaxies hurtling through space. At all size scales between these everything is in motion. Understanding motion is therefore the start of understanding of how the universe around us works. The science of motion and its causes is called Mechanics, which is divided into two main sections: **Kinematics** which describes motion and **Dynamics** which describes the causes of motion. In this chapter, the kinematics of straight-line or one-dimensional motion will be discussed, since it is the simplest type of motion possible. In the next chapter, this discussion will be extended to two-dimensional motion.*

*"To understand motion is to understand nature"*

*Leonardo da Vinci*

## 2.1 POSITION, DISPLACEMENT AND DISTANCE

Consider the following sentence,

"The treasure is hidden 25 paces (steps) northeast of the old hut"

From this sentence, the position of the treasure can be described by a vector which refers to:

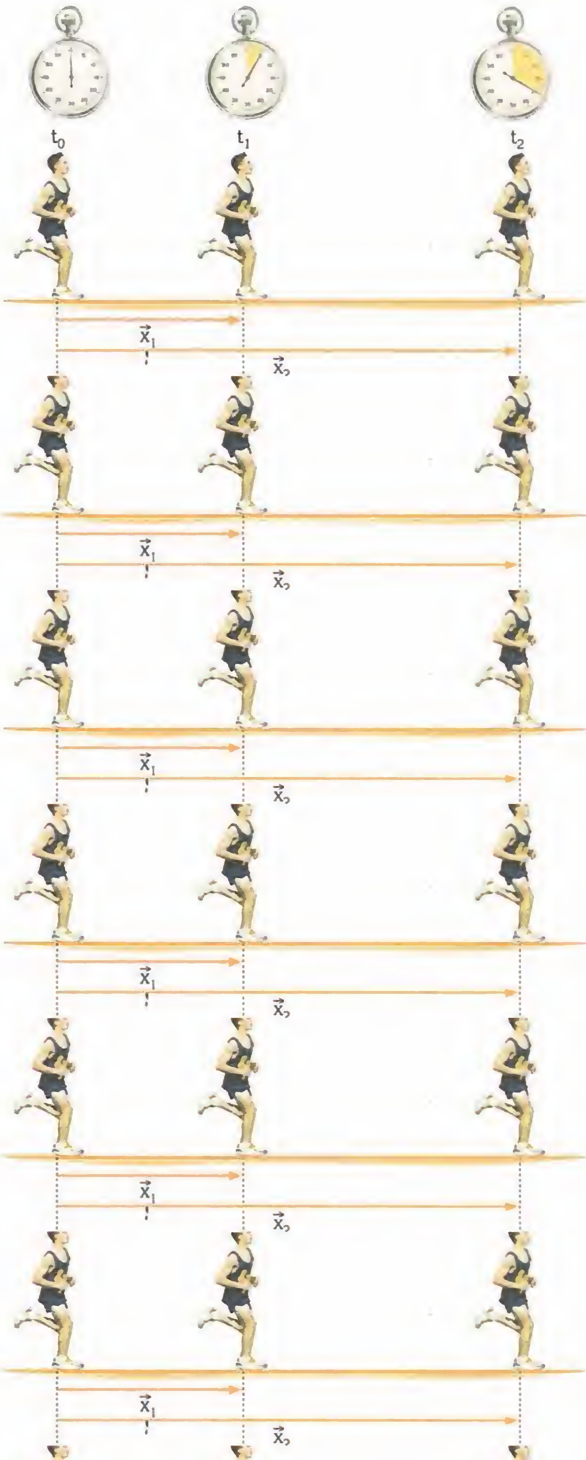
- the distance; 25 paces
- the direction; due northeast
- a chosen reference point; with respect to the old hut

So the **position vector** has a magnitude called distance and a direction relative to a reference point. See Figure 2.1.


Figure 2.1 Vector $\bar{x}$ represents the "position" of the treasure.

An object is said to be in motion when this position vector changes with respect to a reference point. This change in the position of an object is called the **displacement**, which is also a vector quantity.

Consider the straight-line motion of an athlete. In this case, our reference point is the starting point of the athlete at $t_0=0$.



## 2.2 MOTION WITH CONSTANT VELOCITY

Consider a car moving along a straight line in the following diagram:


**Figure 2.4** The positions of a car at times t=0, t=1 s and t=2 s.

The car covers equal displacements in equal time intervals. It moves 10 m every second. This car is said to move with a constant **velocity** which is defined as the displacement per unit time. This is expressed by the following equation:

$$\vec{v} = \frac{\Delta\vec{x}}{\Delta t}$$

Note that the direction of velocity is the same as the direction of displacement. If displacement ($\Delta\vec{x}$) is measured in metres (m) and time ($\Delta t$) is measured in seconds (s), the SI unit of velocity $\vec{v}$ is m/s. The direction of velocity can be defined by positive (+) or negative (–) signs. The magnitude of the velocity vector is called **speed** which is a scalar quantity. Study the table below which describes the speeds of various everyday events.

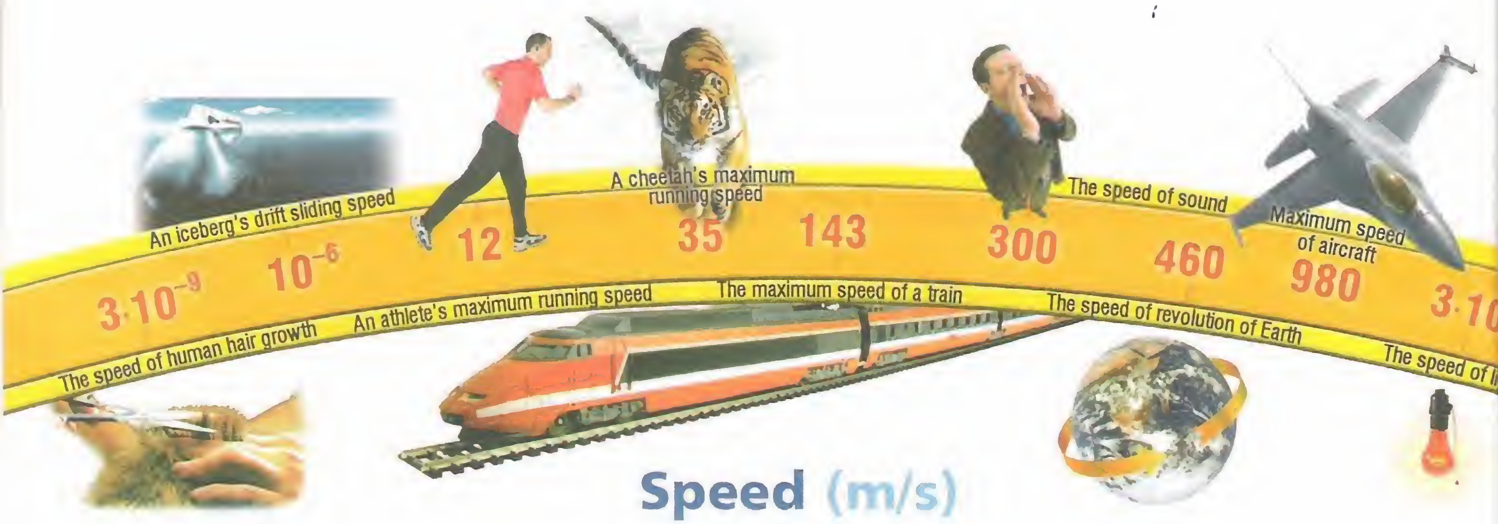



Table 2.1

The position-time graph (sometimes written as x-t graph for simplicity) of the car depicted in Figure 2.4 moving at a steady speed is shown in Figure 2.5.

An important graphical interpretation helps us to derive the constant velocity of the car from the slope of the line:

$$\text{Velocity} = \text{slope of the line in the position–time graph}$$

Mathematically, $$\text{Slope} = \tan\theta = \frac{\Delta x}{\Delta t} = \frac{x_{final} - x_{initial}}{t_{final} - t_{initial}} = v$$

See Figure 2.5. The sign of this slope refers to the direction of velocity, it indicates whether it is in the positive or in the negative direction.

The velocity of the car in the example above is then

$$\text{velocity} = \text{slope} = \frac{\Delta x}{\Delta t} = \frac{40\ \text{m} - 20\ \text{m}}{3\ \text{s} - 1\ \text{s}} = \frac{20\ \text{m}}{2\ \text{s}} = 10\ \text{m/s} \quad \text{in the } +x \text{ direction.}$$

Since the velocity is constant during motion, the velocity-time graph (abbreviated as v-t graph) is a horizontal line, as shown in Figure 2.6.

The position-time graph is used to calculate the velocity, likewise the velocity-time graph can be used to find the magnitude of the displacement:

Since $\Delta x = v\Delta t$ from the defining equation, and $v\Delta t$ equals the shaded area in the v-t graph, as shown in Figure 2.7, it can be concluded that

$$\text{Displacement} = \text{Area between the line and the time axis in a velocity-time graph}$$

For example, during the first 3 s of motion; $\Delta x = (10\ \text{m/s})(3\ \text{s}) = 30\ \text{m}$.

## Position as a Function of Time

The equation for constant velocity as described above can be re-written as

$$\Delta x = v \cdot t \quad \text{where} \quad \Delta x = x_{final} - x_{initial}.$$

Substituting in the expression for $\Delta x$ above, $x_{final} - x_{initial} = v \cdot t$

$$x_{final} = x_{initial} + v \cdot t$$

For clarity $x_{final}$ is denoted as x and $x_{initial}$ as $x_0$,

the final form of the equation is $x = x_0 + v \cdot t$.

This is the position as a function of time, x(t). For example, using SI units, $x(t) = 15 + 40 \cdot t$ means that the initial position of the object is at 15 m, and it moves at a constant velocity of 40 m/s in the positive direction. Similarly $x(t) = 5 - 30t$ means that $x_0 = 5$ m and v=30 m/s in the negative direction.

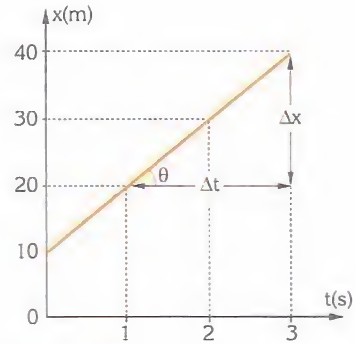


**Figure 2.5** The car has equal displacements in equal time intervals so its position-time graph is a straight line. The slope of the straight line is the constant velocity of the car.

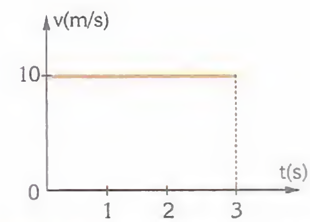


**Figure 2.6** The velocity of the car doesn't change in time. In other words, it is constant.

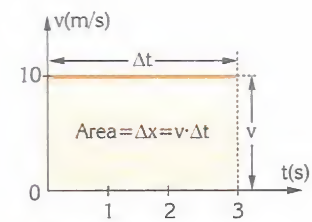


**Figure 2.7** The shaded area gives the displacement.

## Example 2.1

The position of a truck at the moment $t_0=0$ is $x_0=250$ m. It moves with a constant velocity and at $t=20$ s its position is $x=750$ m, as shown in the figure.

a) Calculate the truck's velocity by drawing its position-time graph

b) Draw the velocity-time graph of the truck.

c) Find the position of the truck at the moment $t=30$ s.

d) Find the displacement of the truck between $t=40$ s and $t=60$ s.

e) Write down an equation for the position of the truck as a function of time.

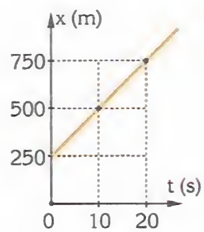

### Solution

a) The truck's position-time graph can be drawn using the information provided in the problem. From the slope of the line, the velocity is;

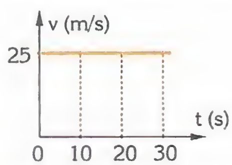
$$\text{slope}=v=\frac{x_f-x_i}{t_f-t_i}$$

$$=\frac{750-250}{20-0}$$

$$v=25 \text{ m/s}$$




b) The velocity of the truck doesn't change in time, therefore its v-t graph must have the form given below:




c) Let the truck's position at $t = 30$ s be x, using the equation

$$\Delta x = v\,\Delta t$$

$$x - x_0 = v(t - t_0)$$

$$x - 250 \text{ m} = 25 \text{ m/s} \cdot (30 \text{ s} - 0)$$

$$x = 1000 \text{ m}$$

d) The truck's displacement between $t=40$ s and $t=60$ s is then

$$\Delta x = v\,\Delta t$$

$$\Delta x = 25 \text{ m/s } (60 \text{ s} - 40 \text{ s})$$

$$\Delta x = 500 \text{ m}$$

e) Since the truck moves at a constant velocity of 25 m/s from $x_0 = 250$ m

$$x = x_0 + vt \qquad \text{thus,} \qquad x = 250 + 25t$$

Figure 2.8 Most objects don't maintain a constant velocity during their motion.

## 2.3 AVERAGE SPEED AND AVERAGE VELOCITY

Imagine how many objects maintain a constant velocity during their motion in everyday life. Usually they don't. The bus shown in Figure 2.8, for example, moving in a straight line, starts its motion from rest, speeds up, moves at a constant speed for a while, then slows down, and finally stops at the next station.

Although the bus doesn't move at a constant velocity all the time, it changes its position between two points along the same line in a given time interval.

This leads to the definition of an **average velocity** from the rate of change in position, which is calculated by the ratio of the displacement $\Delta x$ to the time interval $\Delta t$.

$$\text{Average Velocity} = \frac{\text{Displacement}}{\text{Time Taken}}$$

$$\vec{v}_{av} = \frac{\Delta \vec{x}}{\Delta t} = \frac{\vec{x}_f - \vec{x}_i}{t_f - t_i}$$

Where $\vec{x}_f$ = final position, $\vec{x}_i$ = initial position, $\Delta t$ = time interval

Similarly the **average speed** is calculated as follows:

$$\text{Average Speed} = \frac{\text{Distance Travelled}}{\text{Time Taken}}$$

### Note that

- Both average speed and average velocity have dimensions of Length (L) divided by Time (T), L/T, which is measured in m/s in SI units. Speed is a scalar quantity, whereas, velocity is a vector quantity that also includes the direction of motion.
- Neither average speed nor average velocity provides details of motion during the time interval $\Delta t$. An object may have changed its direction or stopped during its journey.
- Average velocity doesn't depend on the path taken. It is calculated using the total displacement, which is based on the initial and final positions of the path. So, for example, suppose you leave your house in the morning and return in the evening. Your average velocity will be zero, since your displacement is zero.

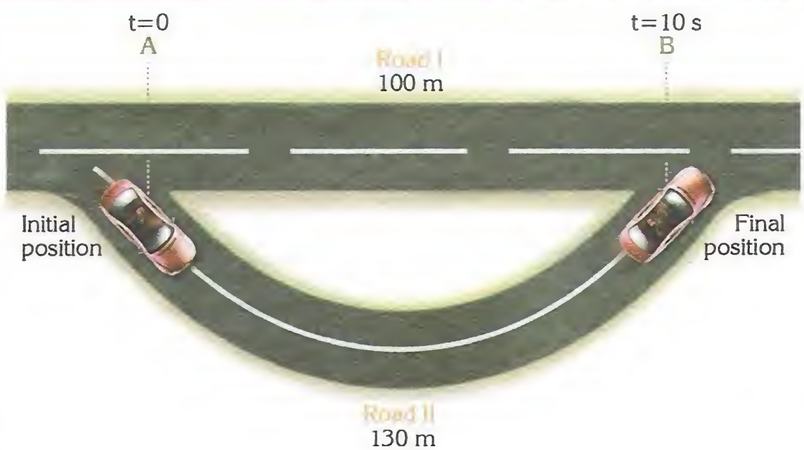
## Example 2.2

Average speed and average velocity

A car is initially at point A. It follows the path shown in the figure and reaches point B after 10 s.

a) Calculate the average speed of the car,

b) Calculate the average velocity of the car.




### Solution

a) Average speed is calculated as the time rate of distance moved, therefore using the length of Road II;

$$v_{av} = \frac{130\text{ m}}{10\text{ s}} = 13\text{ m/s}$$

b) The average velocity is the time rate of change in displacement. The car undergoes a displacement of magnitude 100 m as it moves from point A to B. So it has an average velocity of

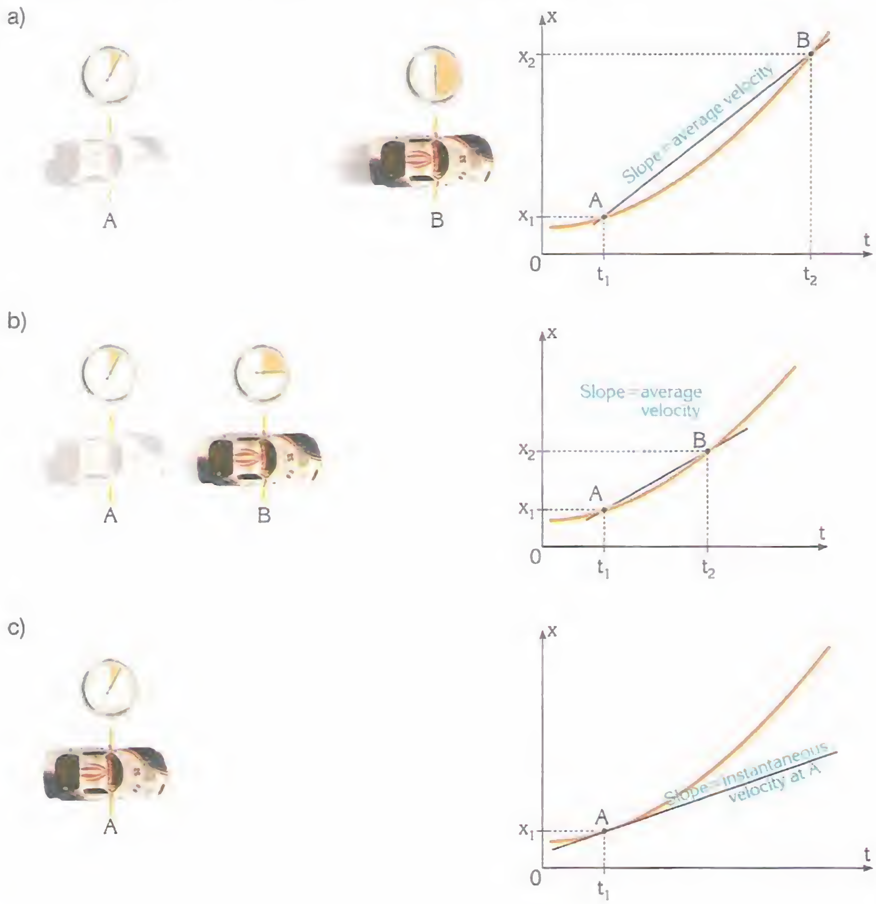
$$v_{av} = \frac{100\text{ m}}{10\text{ s}}$$

$v_{av}$ = 10 m/s to the right along AB.


Figure 2.9 A speedometer reads the instantaneous speed of a car.

## 2.4 INSTANTANEOUS VELOCITY

As was previously mentioned, the average velocity of an object doesn't provide information about how fast and in which direction it was moving at a certain instant of time. The velocity at a single instant of time is called the instantaneous velocity. Think about the speedometer of a car in motion (Figure 2.9). It indicates the magnitude of its instantaneous velocity at each instant of time. In other words, without the direction of travel, it reads the instantaneous speed of the car.




Remember the definition of average velocity calculated over a time interval between two points. Examine the first position-time graph in Figure 2.10.a. The average velocity is equal to the slope of the line which joins the initial and final points A and B on the curve.

Now let's try to calculate the average velocity (slope of the line) using smaller values of $\Delta x$ and $\Delta t$. As point A is approached from point B the $\Delta x$ and $\Delta t$ values get closer to zero and the slope of the line AB approaches the slope of the tangent to the curve at point A. See Figure 2.10.b. The slope of this tangent is the instantaneous velocity of the car at point A. See Figure 2.10.c.

Figure 2.10 As point B approaches point A the slope of the line approaches the slope of the tangent at t1. This is defined as the instantaneous velocity at that instant.

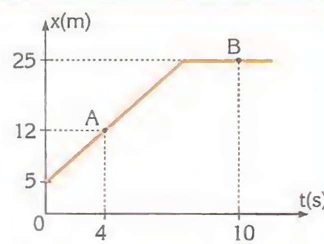
## Example 2.3 Instantaneous velocity

The position-time graph of an object is given in the figure.

a) Calculate the instantaneous velocity of the car at point A when t=4 s.

b) Calculate the instantaneous velocity of the car at point B when t=10 s.

c) Calculate the average velocity between the instants t=0 and t=10 s.



**Solution**

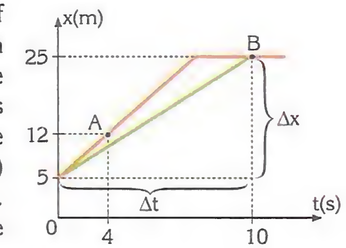
The position-time graph defines the motion of the car: It starts its motion at t=0 s from x=5 m and moves at a constant velocity until x=25 m. After this instant there is no change in displacement, so the car remains stationary for the rest of the motion.

a) The magnitude of the instantaneous velocity of the car at point A is also the constant velocity, which is equal to the slope of the tangent drawn at this point:

$$\text{slope} = v_{ins} = \frac{\Delta x}{\Delta t}$$
$$= \frac{12-5}{4-0} = \frac{7}{4} = 1.75 \text{ m/s}$$

b) When t=10 s, the slope of the line at point B is zero, so is the instantaneous velocity.

c) The average velocity of the car is found from the slope of the line connecting two points on the curve. These points are (t=0, x=5) and (t=10, x=25). Examine the green line in the graph.




The slope of this line is equal to the magnitude of the average velocity of the car between the given points:

$$\text{slope} = v_{av} = \frac{\Delta x}{\Delta t}$$
$$= \frac{25-5}{10-0} = \frac{20}{10} = 2.0 \text{ m/s}$$

## 2.5 ACCELERATION

Consider the linear motion of the motorcycle in Figure 2.11. The timer is started as it passes point A at a velocity of $v_0$= 5 m/s to the right. The motorcycle is observed to constantly increase its velocity by equal amounts of 5 m/s every second. An object which undergoes a uniform change in velocity is said to have a **constant acceleration.**

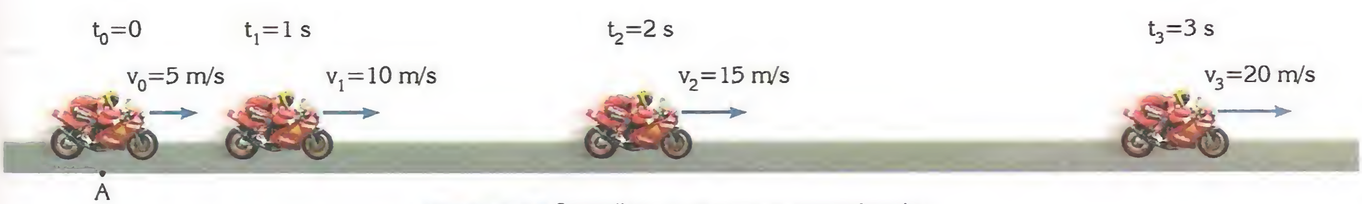



**Figure 2.11** Speeding up at constant acceleration

Now consider the motorcycle in Figure 2.12, as it uniformly slows down, it is said to have a **constant deceleration.**

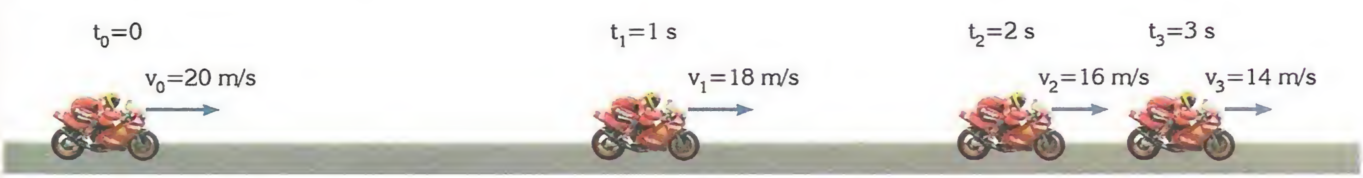



**Figure 2.12** Slowing down at a constant deceleration

Let us now consider the motion of a car along the x-axis in general. At an initial time $t_i$ it has an initial velocity of $v_i$ and at a later time $t_f$ it has a final velocity of $v_f$. See Figure 2.13 below.


**Figure 2.13** Change in velocity results in acceleration.

We define the **acceleration** of the car to be the ratio of the change in velocity to the time interval, that is

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

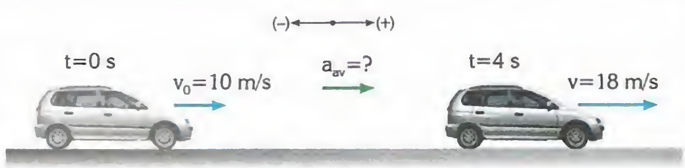
Since acceleration contains the dimensions of $\frac{L}{T^2}$, the SI unit of acceleration is $m/s^2$. Since $\Delta \vec{v}$ is a vector quantity, acceleration is also a vector quantity, and its direction depends on the direction of change in velocity.

## Example 2.4 Motion with constant acceleration

The driver of a car increases his velocity from 10 m/s to 18 m/s in 4 s.

a) Find the acceleration of the car

b) Find the time needed to increase the speed of the car from 18 m/s to 34 m/s, at the same acceleration.




### Solution

a) Using the equation,the acceleration of the car is

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t}$$

$$a = \frac{18-10}{4-0} = 2\,m/s^2$$

b) The time needed to increase the velocity of the vehicle from 18 m/s to 34 m/s is

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t}$$

$$2 = \frac{34-18}{\Delta t} \quad \text{that is} \quad \Delta t = 8\text{ s}$$

## Equations of Motion with Constant Acceleration

Suppose that a car accelerates uniformly from an initial velocity of $\vec{v}_0$ to a final velocity of $\vec{v}$. Let $t_i=0$ and $t_f$ be an arbitrary time t. This is illustrated in Figure 2.14 below.


**Figure 2.14** A car changes its velocity from $v_0$ to v in a time interval of t.

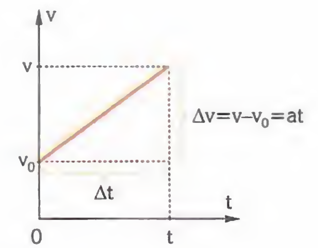



**Figure 2.15** The velocity-time graph of a uniformly accelerating object.

Since the acceleration is constant, there is a linear increase in velocity. When velocity is plotted versus time, the graph is a straight line.This is shown in Figure 1.15.

The slope of this the line in a velocity-time graph provides the constant acceleration.

Mathematically, $$\text{Slope} = \frac{\Delta v}{\Delta t} = a = \frac{v - v_0}{t - 0}$$

Rearranging this equation,

$$a = \frac{v - v_0}{t} \quad \Rightarrow \quad at = v - v_0$$

$v = v_0 + at$ **x-independent velocity equation**

Since it doesn't involve the displacement term, this equation is called the x-independent velocity equation. This equation means that the velocity v at any time t equals the initial velocity $v_0$ at a time t=0 plus the change in velocity "at". Using this equation, velocity is expressed as a function of time. Since acceleration $\vec{a}$ is constant, the velocity changes uniformly in time. Thus, the average velocity is only the average of initial velocity $v_0$ and final velocity v.

$$v_{av} = \frac{v_0 + v}{2} \quad \text{(average velocity equation when a is constant)}$$

Using $\vec{v}_{av} = \frac{\Delta \vec{x}}{\Delta t}$ (average velocity equation for all types of motion)

and replacing $v_{av}$ with $\frac{v_0 + v}{2}$ gives

$$\Delta x = v_{av} \cdot \Delta t \quad \Rightarrow \quad \Delta x = \frac{(v_0 + v)}{2}\Delta t$$

From now on $\Delta x$ will represent the displacement (change in position). The position at a time t=0 will represent the initial position, $x_0$ and the position at a later time t will be represented by x. Therefore $\Delta x$ refers to $x-x_0$. Similarly $\Delta t = t - 0 = t$. In conclusion

$\Delta x = \frac{1}{2}(v_0 + v)t$ **a-independent displacement equation**

Note that, although it doesn't involve an acceleration term, this equation is only valid for motion at constant acceleration.

Substituting $v = v_0 + at$ into the equation $\Delta x = \frac{1}{2}(v_0 + v)t$

$$\Delta x = \left[\frac{v_0 + (v_0 + at)}{2}\right]t \Rightarrow \Delta x = \frac{2v_0t}{2} + \frac{at^2}{2}$$

Rearranging

$\Delta x = v_0t + \frac{1}{2}at^2$ **v-independent displacement equation**

Now we will derive a time-independent equation which will be useful when time information is not provided in problems. We can obtain this equation as follows:

from $v = v_0 + at$ rearranging $\Rightarrow t = \frac{v - v_0}{a}$

by substituting $\frac{v - v_0}{a}$ for t in $\Delta x = \frac{1}{2}(v_0 + v)t$, $\Delta x = \frac{1}{2}(v + v_0)(\frac{v - v_0}{a})$

then $\Delta x = \frac{v^2 - v_0^2}{2a} \Rightarrow v^2 - v_0^2 = 2a\Delta x$

Therefore; $v^2 = v_0^2 + 2a\Delta x$ **t-independent velocity equation**

Table 2.2 lists the set of equations derived for an object moving along a straight line under constant acceleration.

The algebraic signs are used to identify the direction of acceleration and velocity. The possibilities for one dimensional motion are summarized in Table 2.3.

| | |
|---|---|
| $v = v_0 + at$ | x-independent velocity equation |
| $\Delta x = \frac{1}{2}(v_0 + v)t$ | a-independent displacement equation |
| $\Delta x = v_0t + \frac{1}{2}at^2$ | v-independent displacement equation |
| $v^2 = v_0^2 + 2a\Delta x$ | t-independent velocity equation |

**Table 2.2** Equations of motion at constant acceleration in one dimension.

| Algebraic Sign | | Motion and its direction |
|---|---|---|
| Acceleration | Velocity | |
| + | + | Increasing speed in the positive direction |
| – | + | Decreasing speed in the positive direction |
| – | – | Increasing speed in the negative direction |
| + | – | Decreasing speed in the negative direction |

**Table 2.3** The signs of the acceleration and velocity define the direction of one dimensional motion.

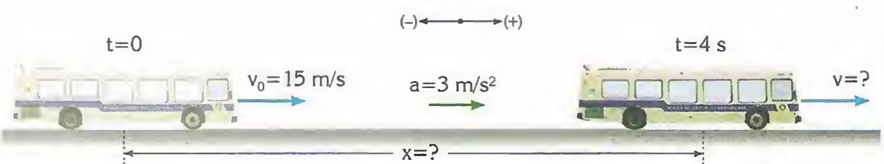
## Example 2.5

Motion with constant acceleration

The driver of a bus which is moving at a constant velocity of 15 m/s, steps on the gas pedal and the bus speeds up with an acceleration of $3 \text{ m/s}^2$ in 4 s. Calculate

a) the velocity of the bus after 4 s.

b) the displacement of the bus in 4 s.



### Solution

a) The velocity reached by the bus in 4 s is found using the equation

$v = v_0 + at = 15\text{ m/s} + (3\text{ m/s}^2)(4\text{ s})$ thus $v = 27\text{ m/s}$

b) Let us assume that the position of the bus is $x_0 = 0$ at the moment $t_0 = 0$. Thus the displacement in 4 s will be $\Delta x = x$ and using the equation

$$x = v_0 t + \frac{1}{2}at^2 = (15\text{ m/s})(4\text{ s}) + \frac{1}{2}(3\text{ m/s}^2)(4\text{ s})^2$$

thus $x = 84\text{ m}$

The displacement can also be obtained from the velocity formula without the time parameter

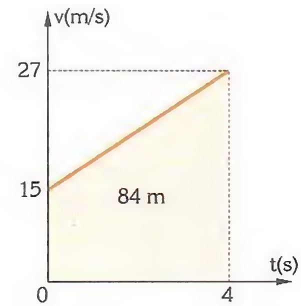
$$v^2 = v_0^2 + 2ax$$

$(27\text{ m/s})^2 = (15\text{ m/s})^2 + 2(3\text{ m/s}^2)x$ thus $x = 84\text{ m}$

Alternatively, the displacement can be found from the area under the velocity-time graph. Here, because the area under the graph is a trapezoid, it is calculated by the equation

$$x = \frac{15+27}{2} \cdot 4$$

$$x = 84\text{ m}$$




Note that the slope of the line in the v-t graph is the acceleration, thus

$$a = \frac{27-15}{4-0} = \frac{12}{4}$$

$$a = 3\text{ m/s}^2$$

## Example 2.6 Acceleration and deceleration

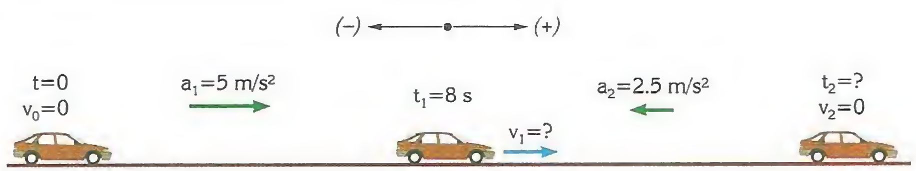
A car waiting at red traffic lights starts moving when the green light is on. For 8 s, the car speeds up with a constant acceleration of 5 m/s$^2$. Afterwards it slows down with a deceleration of 2.5 m/s$^2$ and then stops.

a) Find the velocity of the car at the end of 8 s

b) Find the total time of the motion.

### Solution

Let us study the given data on the motion diagram of the car.




a) As its initial velocity is zero, the velocity it gains after 5 s is found using the equation

$$v_1 = v_0 + at$$

$$v_1 = 0 + (5\text{ m/s}^2)(8\text{ s}) = 40\text{ m/s}$$

b) The deceleration time of the car is found using

$$v_2 = v_1 + at_2$$

$$0 = 40\text{ m/s} + (-2.5\text{ m/s}^2)t_2$$

$$t_2 = 16\text{ s}$$

The Total time of the motion is

$$t = t_1 + t_2$$

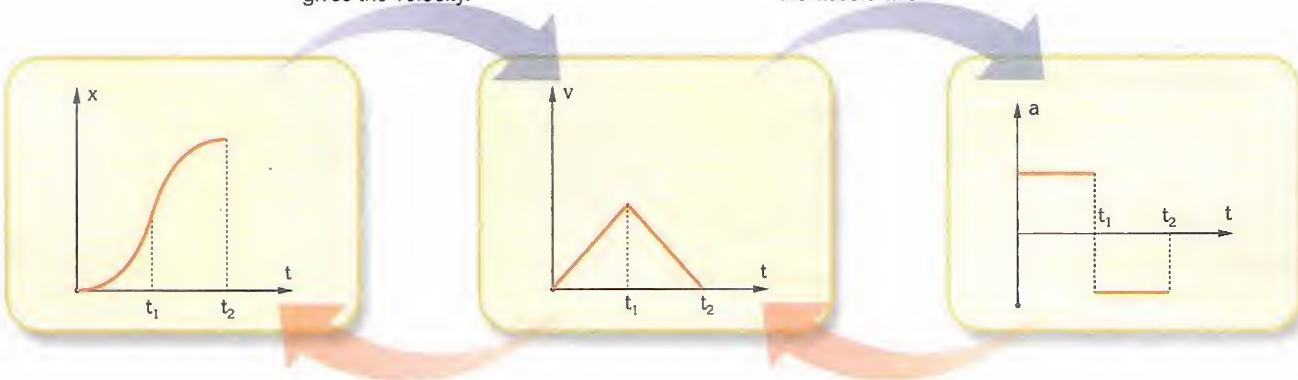
So $t = 8\text{ s} + 16\text{ s} = 24\text{ s}$ later, it stops.

## Graphical Analysis

It might be more convenient to solve motion problems using a graphical method. Here are some of the key points listed to guide you:

a) The slope of the line on a position-time graph gives the velocity.

b) The slope of the line on a velocity-time graph gives the acceleration



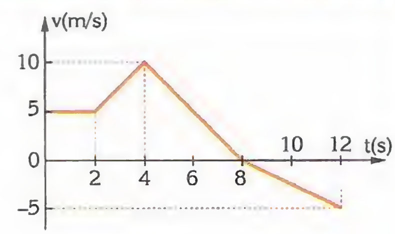

c) The area between the line and the time axis in a velocity-time graph gives the displacement.

d) The area between the line and the time axis in an acceleration-time graph gives the change in velocity

## Example 2.7

The velocity-time graph

The velocity-time graph of a car at a position $x_0 = 5$ m at $t_0 = 0$ is shown in the figure.




a) Find its total displacement in 12 s

b) Find its position at the 12th second.

c) Plot the position-time graph.

d) Plot the acceleration-time graph.

### Solution

a) To find the total displacement of the car, the total area under the graph should be calculated.

The area between t = 0 and t = 2 s is

$$\Delta x_1 = 5 \cdot 2 = 10 \text{ m}$$

The area between t = 2 s and t = 4 s is

$$\Delta x_2 = \frac{5+10}{2} \cdot 2 = 15 \text{ m}$$

The area between t = 4 s and t = 8 s is

$$\Delta x_3 = \frac{1}{2} \cdot 10 \cdot 4 = 20 \text{ m}$$

The area between t = 8 s and t=12 s is

$$\Delta x_4 = \frac{1}{2} \cdot (-5) \cdot 4 = -10 \text{ m}$$

The total displacement is equal to the algebraic sum of these areas.

$$\Delta x = \Delta x_1 + \Delta x_2 + \Delta x_3 + \Delta x_4$$

$$\Delta x = 10 \text{ m} + 15 \text{ m} + 20 \text{ m} - 10 \text{ m}$$

$$\Delta x = 35 \text{ m}$$

b) The car was at position x = 5 m at time t = 0. The displacement of the car was $\Delta x = 35$ m in 12 s, let the position at time t=12 s be $x_4$,

$$x_4 = x_0 + \Delta x = 5 + 35 \quad \text{thus,}$$

$$x_4 = 40 \text{ m}$$

c) To be able to draw the position-time graph of the car, the positions at the instants t=2 s, t=4 s, t=8 s and t=12 s must be known. The positions can be calculated using the $\Delta x$ values which were obtained in part a.

The position at t = 0 is $x_0 = 5$m

The position at t = 2 s is

$$x_1 = x_0 + \Delta x_1 = 5 + 10 = 15 \text{ m}$$

The position at t = 4 s is

$$x_2 = x_1 + \Delta x_2 = 15 + 15 = 30 \text{ m}$$

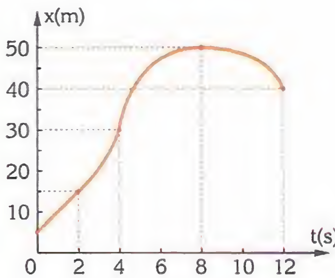
The position at t = 8 s is

$$x_3 = x_2 + \Delta x_3 = 30 + 20 = 50 \text{ m}$$

The position at t = 12 s is

$$x_4 = x_3 + \Delta x_4 = 50 + (-10) = 40 \text{ m}$$

| t(s) | x(m) |
|---|---|
| 0 | 5 |
| 2 | 15 |
| 4 | 30 |
| 8 | 50 |
| 12 | 40 |




d) The slope of the velocity-time graph gives the acceleration. Since the slope between the times 0 – 2 s is zero the acceleration is $a_1 = 0$.

The slope between t = 2 s and t = 4 s is given by

$$a_4 = \frac{10-5}{4-2} = 2.5 \text{ m/s}^2$$

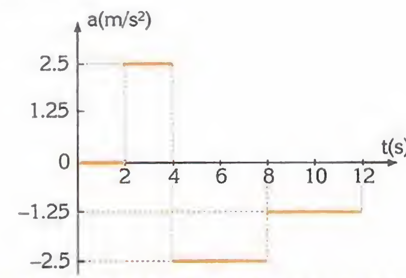
The slope between t = 4 s and t = 8 s is given by

$$a_4 = \frac{0-10}{8-4} = -2.5 \text{ m/s}^2$$

The slope between t = 8 s and t = 12 s is given by

$$a = \frac{-5-0}{12-8} = -1.25 \text{ m/s}^2$$

According to this data, the acceleration-time graph is as shown below.




## 2.6 AVERAGE AND INSTANTANEOUS ACCELERATION

In the previous section, the motion of objects whose velocity changed at a constant rate were discussed, sometimes however, objects which change their velocity non-uniformly may require consideration. In this section, the definition of average and instantaneous acceleration will be extended to include objects moving with a non-constant acceleration.

If an object changes its velocity from $\vec{v}_i$ to $\vec{v}_f$ in a time interval $\Delta t = t_f - t_i$, its average acceleration is calculated by

$$\vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

Figure 2.16 shows the velocity-time graph of the motion. The slope of the line that joins the initial and final velocities gives the average acceleration of the car.

$$\text{Slope} = \frac{\Delta v}{\Delta t} = a_{av}$$

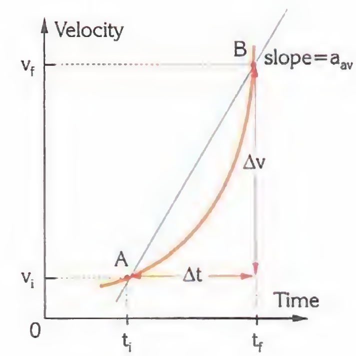



**Figure 2.16** Average acceleration is the slope of the line that joins the initial and final velocities on the v-t graph.

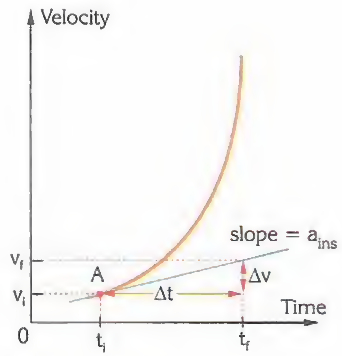



**Figure 2.17** Instantaneous acceleration is the slope of the tangent to the v-t graph.

Average acceleration, doesn't provide details of the motion of a body. The instantaneous accelerations at different moments is required in order to learn more about the motion of the car.

The slope of the tangent to the curve for any specific instant of time is equal to the instantaneous acceleration of the car at that instant. See Figure 2.17.

### Note that

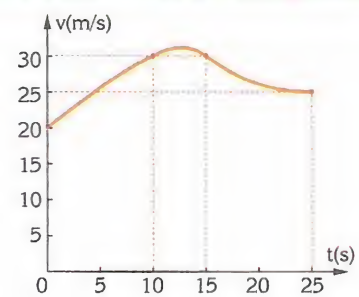
Acceleration is a result of a change either in the magnitude or in the direction of velocity. Consider these examples:

- An object speeding up along a straight line in the same direction. Here the object accelerates since the magnitude of the velocity (speed) changes.
- An object moving in a circle with a constant speed. This is usually called uniform circular motion. Although the magnitude of the velocity is constant, there is a continuous change in its direction. Thus, the object is said to have an acceleration.

## Example 2.8

Velocity-time graph

Find the average acceleration of a car that has a velocity-time graph as shown in the figure between




a) t = 0 and t = 10 s

b) t = 10 s and t = 15 s

b) t = 15 s and t = 25 s

d) t = 0 and t = 25 s

**Solution**

a) From the slope of the line joining the final and initial velocities on the graph, the average acceleration of the car between t = 0 and t = 10 s is given by

$$a_1 = \frac{v_1 - v_0}{t_1 - t_0} = \frac{30-20}{10-0}$$

$$a_1 = 1\,m/s^2$$

b) Between t = 10 s and t = 15 s

$$a_2 = \frac{v_2 - v_1}{t_2 - t_1} = \frac{30-30}{15-10}$$

$$a_2 = 0\,m/s^2$$

c) Between t = 15 s and t = 25 s

$$a_3 = \frac{v_3 - v_2}{t_3 - t_2} = \frac{25-30}{25-15}$$

$$a_3 = -0.5\,m/s^2$$

d) Between t = 0 and t = 25 s

$$a_4 = \frac{v_4 - v_0}{t_4 - t_0} = \frac{25-20}{25-0}$$

$$a_4 = 0.2\,m/s^2$$

## 2.7 FREELY FALLING OBJECTS

A common example of one-dimensional motion with constant acceleration is that of an object falling freely towards the Earth. If air resistance is neglected as well as any small variations in gravity with altitude, all objects, regardless of their mass and size, fall with the same acceleration towards the centre of the Earth. This constant "acceleration of free-fall" is denoted by g, which has the value 9.8 m/s$^2$. The motion of an object in a vertical line (rising as well as falling) with a constant acceleration of g in the absence of air resistance is called **free-fall**. Compare the motion of a rock and feather in the presence and absence of air resistance in Figure 2.18.

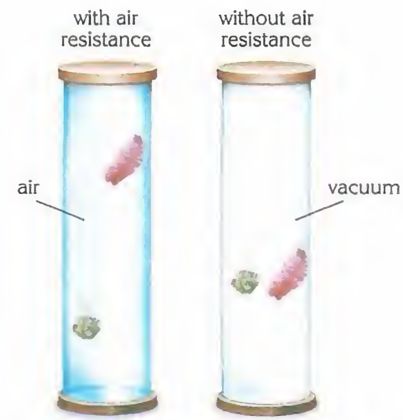



**Figure 2.18** The positions of the rock and the feather are shown at a given instant as they fall freely in the absence and the presence of air. There is more resistance acting on the feather, so the stone falls earlier in the first tube. However, they fall at the same rate in the absence of air resistance in the second tube.

The equations (Table 2.2) derived in the previous section for constant acceleration are all valid for free-fall. Simply replace $\Delta x$ by $\Delta y$ (because motion takes place in the vertical direction), also g is now the acceleration, a. (Table 2.4)

If an object is released from rest which means $v_0=0$, the equations in Table 2.4 are simplified, as shown in Table 2.5. The motion diagram in Figure 2.19 shows the change in the velocity and height of a tennis ball during free-fall.

| |
|---|
| $v = v_0 + gt$ |
| $\Delta y = (\frac{v_0 + v}{2})t$ |
| $\Delta y = v_0 t + \frac{1}{2}gt^2$ |
| $v^2 = v_0^2 + 2g\Delta y$ |

**Table 2.4** Equations of motion in the vertical direction under the effect of the constant acceleration of free fall.

| |
|---|
| $v = gt$ |
| $\Delta y = \frac{1}{2}vt$ |
| $\Delta y = \frac{1}{2}gt^2$ |
| $v^2 = 2g\Delta y$ |

**Table 2.5** Free fall equations for $v_0=0$

Remember that the direction of the initial velocity doesn't change the nature of free-fall. Objects thrown upwards, thrown downwards, or released from rest all experience the same constant acceleration. The equations in the table are valid for each case.

In problem solving, when these equations are applied it is more convenient to choose the direction of the initial velocity as the reference direction for g, v, and $\Delta y$. The direction of the initial velocity may be taken as positive, however one must then be consistent with other all the quantities according to this choice. That is, if an object is thrown downwards or released from rest, the sign of g must be taken as positive (+). If it is thrown upwards, g must be taken as negative (–).

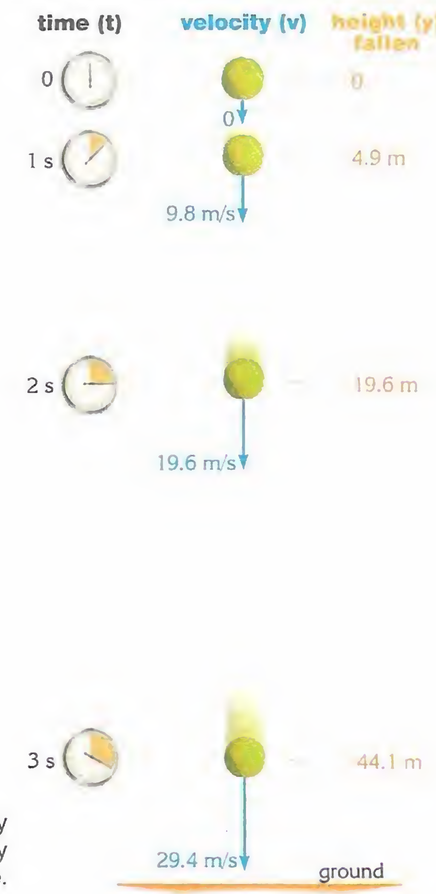



**Figure 2.19** A tennis ball falling freely in a vacuum. Notice how its velocity and height change with time.

## Example 2.9 — Falling freely from rest

A stone is dropped from the top of the Galata Tower, which is 68 m high. (Assume that there is no air resistance and $g=9.8\ m/s^2$)

a) How far will it have fallen after 3.0 s?

b) At what speed does the stone hit the ground?

**Solution**

a) Using the equation

$$\Delta y = \frac{1}{2}gt^2 = \frac{1}{2}(9.8\ m/s^2)(3.0\ s)^2 = 44\ m$$

b) Using the equation

$$v^2 = 2g\Delta y$$

$$v^2 = 2 \cdot 9{,}8 \cdot 68 \qquad v = 36{,}5\ m/s$$

## Example 2.10 — Free fall from rest

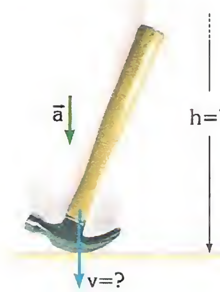
A worker who is on scaffolding drops a hammer he is holding in his hand. If the hammer falls to the ground in 2 s, find

a) the height fallen by the hammer.

b) the velocity of the hammer as it strikes the ground.

c) the velocity and the height of the hammer from the ground at time t = 1 s.

(Take $g=10\ m/s^2$ in this and following examples.)




**Solution**

a) Choosing the downward direction of motion to be positive. The height, h, of the worker from the ground is equal to the displacement covered by the hammer. The initial velocity of the hammer is $v_0 = 0$, using the equation,

$$\Delta y = h = \frac{1}{2}gt^2 \quad h = \frac{1}{2}\cdot(10\ m/s^2)(2\ s)^2 \quad \text{thus } h = 20\ m$$

b) The velocity of the hammer when it strikes the ground is found using

$$v = gt = (10\ m/s^2)(2\ s)$$

$$v = 20\ m/s \text{ downwards}$$

c) The velocity of the hammer at t=1s is found using

$$v = gt = (10\ m/s^2)(1\ s)$$

$$v = 10\ m/s \quad \text{downwards}$$

And its displacement is found using

$$\Delta y = \frac{1}{2}gt^2 \quad \Delta y = \frac{1}{2}(10\ m/s^2)(1\ s)^2$$

thus $\Delta y = 5$ m downwards

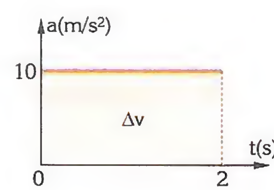
If the height of the hammer from the ground at this moment is $h_2$,

$h_2 = 20\ m - 5\ m = 15\ m$

**Solution 2**

Parts b) and a) can also be solved using the graphical method.

b) The area under the acceleration-time graph of the hammer gives the change in the velocity. Using

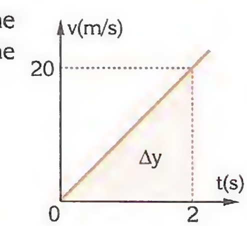



$$\text{Area} = \Delta v = at$$

$$v - 0 = (10\ m/s^2)(2\ s)$$

$$v = 20\ m/s$$

a) The area under the velocity-time graph of the hammer gives the displacement. Therefore



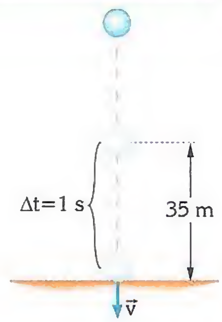

$$\text{area} = \Delta y = y - 0$$

$$= \frac{1}{2}\cdot(20\ m/s)(2\ s)$$

$$y = h = 20\ m$$

## Example 2.11 Object falling freely from rest

The ball shown in the figure, is released from rest and it travels 35 m in its last second of motion. Calculate the height from which the ball was released.




### Solution

Taking the velocity of the ball as $\vec{v}_1$, 1 s before striking the ground. Let the downward direction of motion be positive. Using the equation for displacement in the last second of the ball's motion

$$\Delta y = v_1 \cdot t + \frac{1}{2}gt^2$$

$$35\ m = v_1 \cdot 1 + \frac{1}{2} \cdot (10\ m/s^2)(1\ s)^2$$

$$v_1 = 30\ m/s$$

The time at which the velocity reaches 30 m/s is found using the equation,

$$v_1 = gt$$

$$30\ m/s = (10\ m/s^2)t \quad \text{thus} \quad t = 3\ s$$

The total time of motion is

$$t = 3\ s + 1\ s = 4\ s$$

Therefore the height of the ball is found using the equation

$$\Delta y = h = \frac{1}{2}gt^2$$

$$h = \frac{1}{2} \cdot (10\ m/s^2)(4\ s)^2$$

$$h = 80\ m$$

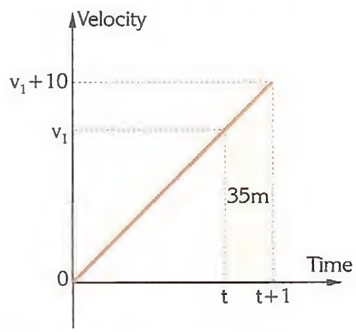
### Solution 2

This problem can also be solved using the velocity-time graph of the ball. If the velocity 1 s before hitting the ground is $v_1$ and the velocity on striking the ground is $v_2$, using the equation

$$v_2 = v_1 + gt$$

$$v_2 = v_1 + (10\ m/s^2)(1\ s) \quad \text{thus} \quad v_2 = v_1 + 10$$

Let the ball reach the velocity $v_1$, t seconds after it falls freely, its velocity-time graph will be as shown below.




The shaded area under the graph is,

$$\Delta y = \frac{(v_1 + v_1 + 10)}{2} \cdot (t + 1 - t)$$

$$35\ m/s = \frac{2v_1 + 10\ m/s}{2} \cdot 1$$

$$v_1 = 30\ m/s$$

The time at which the ball reached a velocity of 30 m/s, is found from the slope of the graph,

$$\text{Using slope} = g = \frac{v - v_0}{t - t_0}$$

$$10 = \frac{30 - 0}{t - 0}$$

$$t = 3\ s$$

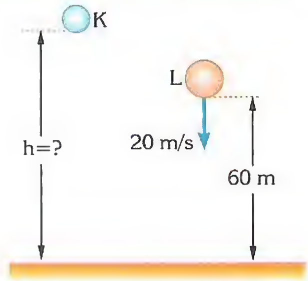
Since the area between the time axis and the curve until the moment t+1 gives the total displacement of the ball, using the equation

$$h = \frac{1}{2}g(t+1)^2$$

$$h = \frac{1}{2} \cdot (10\ m/s^2)(4\ s)^2$$

$$h = 80\ m$$

## Example 2.12
Object thrown vertically downwards




A ball L is thrown downwards from a height of 60 m with a velocity of 20 m/s and a ball K is released from rest at a height h. They both strike the ground at the same velocity. What is the height h?

**Solution**

Taking the downward direction to be positive, the velocity of ball *L* at the moment it strikes the ground after travelling a distance of *60 m* is found using the equation

$$v^2 = v_0^2+2g\Delta y$$

$$v^2 = (20)^2 + 2(10\ m/s^2)(60\ s)^2$$

$$v = 40\ m/s \quad \text{downwards}$$

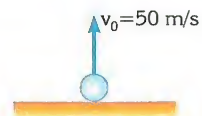
*K* hits the ground at the same velocity as L, height h is found using the equation

$$v^2 = v_0^2 + 2g\Delta y$$

$$(40\ m/s)^2 = 0 + 2(10\ m/s^2)h$$

$$h = 80\ m$$

## Example 2.13
Object thrown vertically upwards




A ball is thrown upwards with a velocity of 50 m/s, as shown in the figure. Find

a) The maximum height that it can reach.

b) The time to reach its maximum height.

c) Its height from the ground and its velocity at the instants t = 3 s and t = 7 s.

d) Its velocity at the moment it strikes the ground. (Take $g = 10\ m/s^2$)

**Solution**

a) This problem can be solved by choosing the upward direction to be positive, since it is the direction of the initial velocity.

Hence the ball's velocity at maximum height will be zero, using the equation

$$v^2 = v_0^2 + 2g\Delta y$$

$$0 = (50\ m/s)^2 + 2(-10\ m/s^2)h_{max}$$

$$h_{max} = 125\ m$$

b) The time to reach its maximum height is found using the equation

$$v = v_0 + gt$$

$$0 = 50\ m/s + (-10\ m/s^2)t$$

$$t = 5\ s$$

c) The velocity of the ball at t= 3 s is found using the equation

$$v = v_0 + gt$$

$$v = 50\ m/s + (-10\ m/s^2) \cdot 3$$

$$v = 20\ m/s$$

The height at any moment of this motion is equal to its displacement. Therefore the ball's height is found using the equation

$$\Delta y = v_0 \cdot t + \frac{1}{2}gt^2$$

$$h = (50\ m/s)(3\ s) + \frac{1}{2}\cdot(-$$

$$h = 105\ m$$

Its velocity at t = 7 s is found using the equation

$$v = v_0 + gt$$

$$v = 50\ m/s + (-10\ m/s^2)(7\ s) \quad \text{thus} \quad v = -20\ m/s$$

The negative sign on the velocity indicates that the ball is moving downwards at this moment.

Its height from the ground at time $t = 7$ s is found using the equation

$$\Delta y = v_0 \cdot t + \frac{1}{2}gt^2$$

$$h = 50 \cdot 7 + \frac{1}{2} \cdot (-10) \cdot 7^2 \quad \text{thus} \quad h = 105 \text{ m}$$

Note that when the object is 105 m above the ground (when t=3 s and t=7 s ), the velocities are equal in magnitude (the same speed) but opposite in direction.

d) The total time of flight is twice the time it takes the ball to reach the highest point on its path, therefore

$$t_{flight} = 2 \cdot 5 = 10 \text{ s}$$

The velocity just before it strikes the ground can be found from

$$v = v_0 + gt$$

$$v = 50 + (-10) \cdot 10 = -50 \text{ m/s}$$

(the negative sign of the velocity means that it is travelling in the downward direction)

## Example 2.14 — Object thrown upwards from a height

A stone is thrown vertically upwards from a 25 m high tower at a velocity of 20 m/s. After reaching its maximum height, the stone falls downwards, without hitting the tower, along the same trajectory, and then strikes the ground. Find

a) The maximum height it reaches from the point it was thrown.

b) The time taken for the stone to reach its maximum height.

c) During the downward part of its trajectory, find its velocity when it passes its starting point.

d) The total time of flight.

e) Its velocity at the instant it strikes the ground.


### Solution

a) Choose the direction of its initial velocity (upwards) to be positive. Since the velocity at maximum height is zero, using the equation

$$v^2 = v_0^2 + 2g\Delta y$$

$$0 = 20^2 + 2 \cdot (-10)\, h_{max} \quad \text{thus } h_{max} = 20 \text{ m}$$

b) Since the velocity at the top is zero, the time taken to reach the maximum height is found using the equation

$$v = v_0 + gt$$

$$0 = 20 + (-10)\, t_{rise} \quad \text{thus} \quad t_{rise} = 2 \text{ s}$$

c) The time taken for the stone to reach its maximum height is equal to the time taken to fall to its initial position. Therefore, the time taken to return to its initial position is

$$t = 2t_{rise}$$

$$t = 2 \cdot 2 = 4 \text{ s}$$

Using this value of t in the equation

$$v = v_0 + gt$$

$$v = 20 + (-10) \cdot 4 \quad \text{thus} \quad v = -20 \text{ m/s}$$

d) The ball is 25 m below its starting point at the end of its flight. This means $\Delta y = -25$ m. Applying the equation

$$\Delta y = v_0 t + \frac{1}{2}gt^2$$

$$-25 = 20 \cdot t_{flight} + \frac{1}{2}(-10)t_{flight}^2$$

$$50t^2 - 20t - 25 = 0 \quad \Rightarrow \quad t^2 - 4t - 5 = 0$$

The roots of this equation are

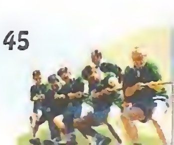
$$(t-5)(t+1) = 0$$

$$t = 5 \quad \text{and} \quad t = -1$$

Since time cannot be negative

$$t_{flight} = 5 \text{ s.}$$



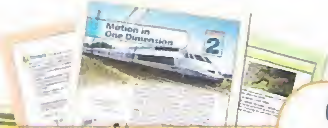


e) Its velocity just before it strikes the ground is then

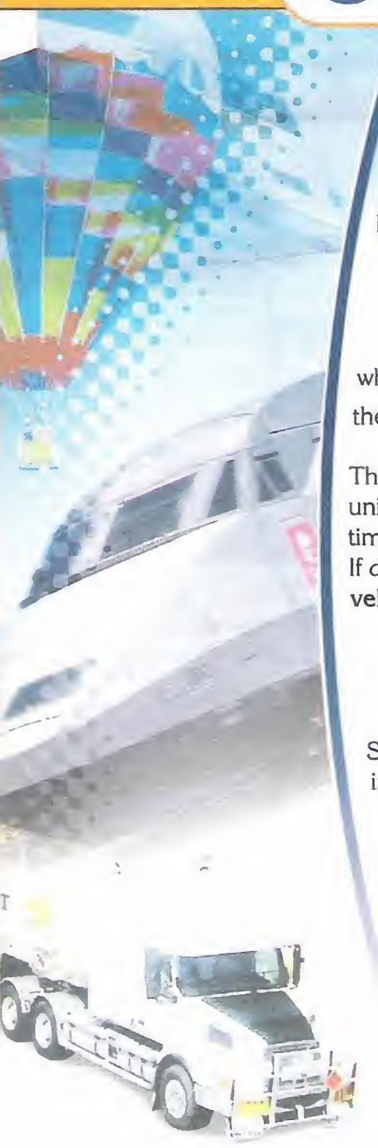
$$v = v_0 + gt$$

$$v = 20 + (-10) \cdot 5$$

$$v = -30 \text{ m/s}$$



# Summary



The position of an object is described by a position vector which describes its distance and direction relative to a reference point.

The change in the position of an object is called the displacement which is also a vector quantity. Displacement is defined by

$$\Delta\vec{x} = \vec{x}_2 - \vec{x}_1$$

where $\vec{x}_2$ is the position vector at time $t_2$ and $\vec{x}_1$ is the position vector at time $t_1$.

The rate of displacement (change in position per unit time) is called velocity. If this rate is constant in time, the motion is said to have constant velocity. If displacement is divided by total time the **average velocity** of an object is obtained:

$$\text{Average Velocity} = \frac{\text{Displacement}}{\text{Time Taken}}$$

Similarly we calculate the **average speed** which is independent of the direction of motion as:

$$\text{Average Speed} = \frac{\text{Distance Travelled}}{\text{Time Taken}}$$

The velocity of an object at a certain instant of time is called the **instantaneous velocity**.

The change in velocity per unit time is called the acceleration.

$$\vec{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

If an object undergoes a uniform change in velocity with time it is said to have a constant acceleration.

The following equations of kinematics are used to define uniformly accelerated motion:

where

a is the constant acceleration

$v_0$ is the initial velocity

v is the final velocity

x is the displacement

t is the time

| |
|---|
| $v = v_0 + at$ |
| $\Delta x = v_0 t + \frac{1}{2}at^2$ |
| $\Delta y = \frac{1}{2}(v_0 + v)t$ |
| $v^2 = v_0^2 + 2a\Delta x$ |

Some graphical methods are also used, these are:

The slope of the line of the position-time graph which gives the velocity.

The area under the velocity-time graph, which gives the displacement.

The slope of the line of the velocity-time graph which gives the acceleration.

The area under the acceleration-time graph, which gives the change in velocity.

If air resistance is neglected, all objects fall with the same acceleration of free fall, 9.8 $m/s^2$, which is constant and directed downwards towards the centre of the Earth throughout the motion. The equations for constant acceleration above are also valid for free fall. Only $\Delta x$ is replaced by $\Delta y$ and a is replaced by g in the equations.

# QUESTIONS AND PROBLEMS

### 2.1 Position, Displacement and Distance

1. Explain the difference between displacement and distance.

2. What do we pay for when we travel by taxi? Distance or displacement? What about when we travel on an airplane?

3. A mountaineer and a slope parachuter start their motion from the top of a hill at the same time. Discuss their respective displacements and distances, if they end up at the same point on the land.

4. a) Discuss whether distance can be smaller than displacement.

   b) Does the kilometre indicator in a car display its distance or displacement?

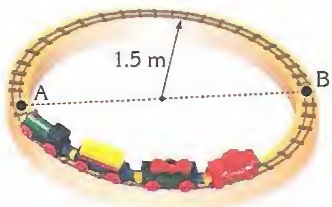
5. A helicopter takes off and travels 300 km due north, then returns to its landing field. What is its displacement?

6. A particle is initially at a position of $x_i = -4$ m relative to the origin. It then moves to the right along the x-axis and finally stops at a position of $x_f = 8$ m.

   a) By drawing an x-axis, show its initial and final position vectors.

   b) Find its displacement.

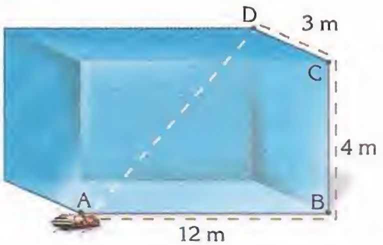
7. A toy train moves along a circular railway from point A to B, as shown in the figure. If the radius of this path is 1.5 m




   a) Calculate the distance moved ($\pi = 3$)

   b) Calculate the magnitude of the displacement

8. 




An insect in a corridor, measuring 12 m x 4 m x 3 m, walks along the path ABCD shown in the diagram. What distance would it travel if it flew directly from point A to point D along a straight line? (In other words, find its displacement)

### 2.2 Motion with constant velocity

9. Convert the following units

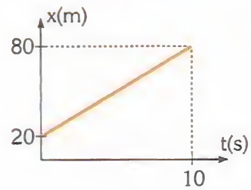
   a) 72 km/h into m/s

   b) 27 cm/s into m/s

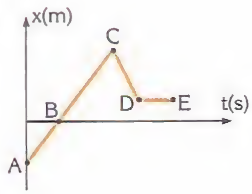
   c) 15 m/s into km/h

10. What does the slope of a line in a position-time graph give?

**11.** The position-time graph of a car moving along the x-axis is given in the figure. What is the speed of the car in m/s?




**12.** According to the x-t graph on the right in which direction (positive or negative) is the object moving between points




a) A and B

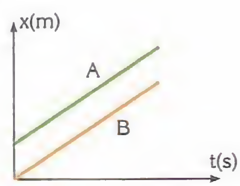
b) B and C

c) C and D

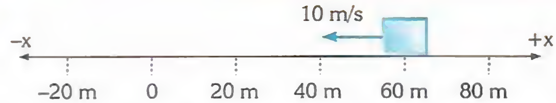
d) D and E

**13.** A car starts from a position of x=120 m and moves along the x axis in a positive direction at a constant speed of v=8 m/s. Draw a position time graph for the car over a time period of 10 s.

**14.** The position-time graphs of two objects A and B, are shown in the figure. Which object is faster?

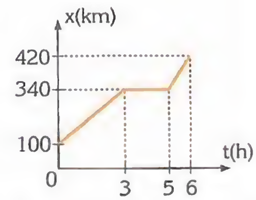



**15.**




A diagram of an object moving along the x-axis taken at a time t=0 is given above. Draw the position-time graph of the object for the next 7 seconds, if the velocity of the object is 10 m/s in the given direction and constant.

**16.** The position-time graph of a car moving along the x-axis is given in the right-hand diagram.




a) Draw the velocity-time graph.

b) What is the displacement of the car in the first 5 hours?

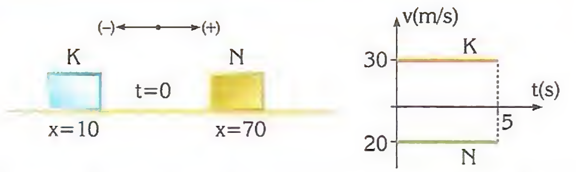
**17.** In astronomy the light-year is used as a unit of distance. It is the distance that light can travel in one year. (For instance, our galaxy, the Milky Way, is about 150 000 light-years across). If light travels at a constant speed of 300 000 km/s,

a) Calculate 1 light-year in km.

b) Calculate how many years it takes for the light to reach us from our closest star (Proxima Centauri) which is about 4.0 x $10^{13}$ km away.

c) Logically, Proxima Centauri is the first destination for interstellar travel. If the fastest man-made spacecraft, the Helios II, has a speed record of 70.2 km/s, how long would this journey take?

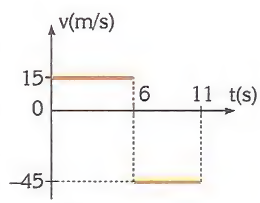
**18.**



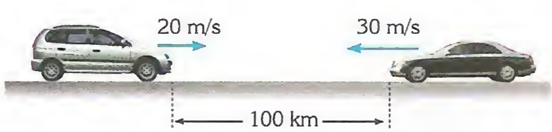

The positions of cars K and N at a time t=0 and their respective v-t graphs are shown in the figure above. What is the distance between the cars at a time t=5 s?

**19.** a) Draw an x-t graph representing the v-t graph in the figure to the right, by taking the starting position as $x_0=0$.

b) Draw the same graph for an object starting from $x_0=20$ m.




**20.**



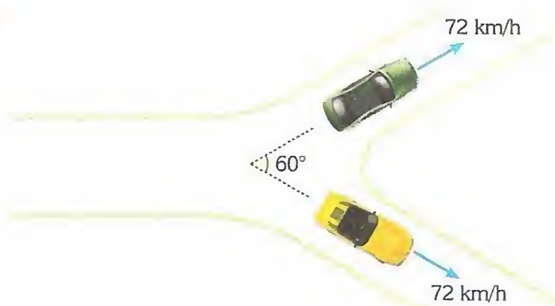

Two vehicles move in opposite directions to each other with constant speeds of 20 m/s and 30 m/s, as shown in the figure. If the initial distance between the vehicles is 100 km, how many seconds later do they meet?

**21.** A car travels from city A to city B at a constant speed of 60 km/h. A bus leaves city A 1h after the car and catches it as it enters city B. The distance between the cities is 240 km. What is the speed of the bus (assumed to be constant) during its journey?

**22.** Assume that Serkan and Yeşim can run at constant speeds of 8 m/s and 5 m/s, respectively. They finish a 400 m race together, since Serkan starts after a time interval $\Delta t$ seconds later than Yeşim. Find the time interval $\Delta t$.

**23.** A police car moving at 140 km/h is chasing a lorry moving at 120 km/h. If the initial distance between them is 500 m, how many minutes later and at what distance from its initial position does the police car catch the lorry?

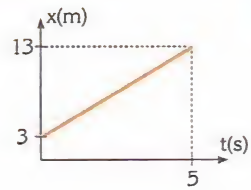
**24.**




Two cars are travelling on a road, which splits at an angle of 60°, both have the same speed of 72 km/h. After they split at the junction what is the distance between them 15 minutes later?

**25.** How long does it take a train of length 120 m moving at a constant speed of 36 km/h to completely passover a bridge of length 200 m?

26. The position-time graph for an object is given in the right-hand diagram. Write the equation for its position as a function of time, x(t).




27. Two objects travel according to the equations:

$x_1(t) = 600 + 20\,t$ (in metres)

$x_2(t) = 1400 - 30\,t$ (in metres)

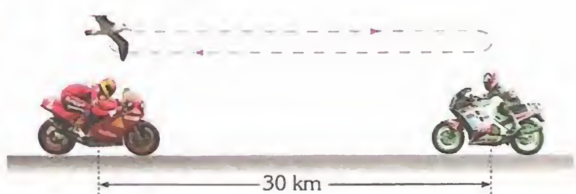
a) What is the velocity of each object?

b) What is the initial position (at t=0) of each object?

c) Determine where (x=?) and when (t=?) they meet?

(Hint: One way of finding their common meeting point is by setting $x_1=x_2$ and solving the equation for t)

28.




Two motorcyclists are moving towards each other at a constant velocity of 15 km/h. When the distance between them is 30 km a bird, above the red motorcyclist, starts to fly at a velocity of 25 km/h towards the blue motorcyclist, as shown in the figure. As soon as the bird reaches the blue motorcyclist, it turns back and flies towards the red motorcyclist at the same speed. After flying backwards and forwards between the two motorcyclist in this way, how many km of distance will the bird have flown when the motorcyclists meet?

## 2.3 Average Speed, Average Velocity, 2.4 Instantaneous Velocity

29. Can an object have a constant speed but changing velocity?

30. Can the velocity of a car which is turning a curve be held constant?

31. In which type of motion is the average velocity of an object equal to its instantaneous velocity?

32.


The cyclist in the figure starts cycling at point A. He reaches point C in 80 s, returns and comes to rest at point B in 20 s. Calculate the cyclist's

a) Average speed and average velocity in 80 s.

b) Average speed and average velocity in 100 s.

c) Average speed and average velocity, if the cyclist does not stop at point B but returns again to point A, from point C, in a total time of 100 s.

**33.** An object is thrown vertically upwards. After it reaches a maximum height, it falls back to its initial position. What can you say about its average velocity?

**34.** In big cities, traffic lights occurring in series on a street are programmed according to a given average car speed. In this system called green wave, cars passing a green light, and moving with this average speed between two traffic lights continue to meet green. Let's say the distance between two successive traffic lights is 900 m. If the time difference between green lights is 1 minute, what is the average speed of a car which continues meeting green lights?

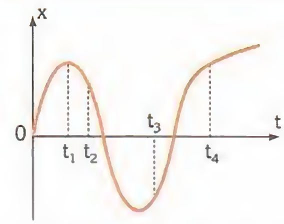
**35.**

In 1991, Carl Lewis set a new world record in the 100 m race with a time of 9.86 s.

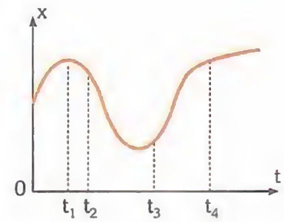
a) Calculate his average velocity.

b) If a cheetah, the fastest of all land animals, with an average speed of 29 m/s took place in the same race, how many seconds before Lewis would it finish the race?

**36.** a) The position - time graph for an object moving on the x-axis is given in the figure. Determine whether the velocity is positive, negative or zero (whether the object moves to the right, to the left, or is at rest) at the times given in the graph.




b) Answer the same questions in part (a) for the graph given on the right. Is your answer any different? Why or why not? What is the difference between the two graphs?

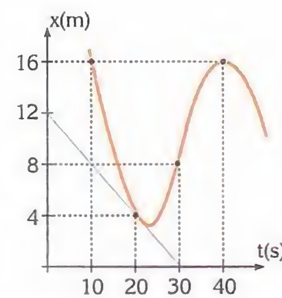



**37.** The position-time graph for an object moving in one dimension is given on the right.

a) Find the average velocity in a time interval from t=10 s to t=30 s.

b) Find the instantaneous velocity at time t=20 s.

c) What is its velocity at t=40 s?




**38.** Find the magnitude of average velocity

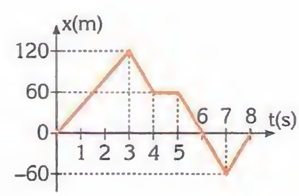
a) From 0 to 3 s

b) From 0 to 4 s

c) From 3 to 5 s

d) From 3 to 7 s

e) From 0 to 8 s



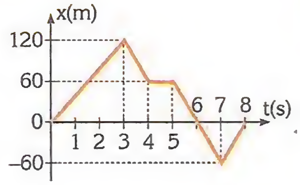
39. Find the instantaneous velocity

a) At 2.5 s

b) At 3.5 s

c) At 4.5 s

d) At 7.5 s




40. A car travels eastwards at 60 km/h for 2 h, then travels northwards at 20 km/h for 8 h.

a) Find the average speed of the car.

b) Find the average velocity of the car.

41. A car travels at 60 km/h for 2 h, then stops for 2.5 h, and then it travels 80 km in 30 minutes, all in the same direction.

a) Find its average velocity and its average speed.

b) Plot its x-t graph (take $x_0=0$)

42. In the first half of his journey, an automobile moves with a velocity of $v_1=4$ m/s, and in the second half of his journey with a velocity of $v_2=12$ m/s, along a straight road.

a) Find the average velocity.

b) Show that the average velocity is $\frac{2v_1v_2}{v_1+v_2}$ and that it is less than the arithmetical average of $v_1$ and $v_2$.

43. A student walks to school from his home with a velocity of 6 m/s and returns home with a velocity of 4 m/s. Find the student's

a) Average speed

b) Average velocity.

## 2.5 Acceleration

44. a) Describe a situation where an object has zero acceleration but non-zero velocity?

b) Describe a situation where an object has zero velocity but non-zero acceleration?

45. A car is moving eastward but accelerating westward. Is this possible? How?

46. A car has a positive acceleration and a negative velocity at a given instant. In which direction is it moving at that instant?

47. A car accelerates from rest to 108 km/h in 6 seconds. What is the acceleration of the car?

48. A particle has a velocity $v_0=48$ m/s at t=0. Between t=0 and t=12 s, the velocity decreases uniformly to zero. What is the acceleration of the particle? What is the significance of the sign in your answer?

**49.** A train travelling on a straight track has a velocity of v=6 m/s at t=0. Its velocity increases uniformly and reaches a value of v=18 m/s at t=8 s. What is its acceleration?

**50.** An object moving with a velocity of $v_0$=20 m/s in the positive direction starts to decelerate uniformly at t=0. At t=8 s, its velocity becomes v=16.8 m/s.

a) Find its acceleration

b) Find its velocity at t=40 s.

c) How long will it take for the object to stop?

### Equations of Motion with Constant Acceleration

**51.** A car starts from rest and accelerates at 1.5 m/s$^2$. Find its velocity and displacement 10 s later.

**52.** A train travelling at 54 km/h starts to decelerate at 0.3 m/s$^2$.

a) How long will it take to stop?

b) What is the stopping distance?

**53.** The maximum acceleration of a train is given as 0.2 m/s$^2$.

a) Find the velocity and displacement of the train in 12 s if it starts to speed-up from rest.

b) How long will it take for this train to reach a velocity of 54 km/h if it continues to accelerate at the same rate?

**54.** The velocity of an object at t=0 is measured as v=25 m/s. Find its speed and displacement at t=10 s

a) if it accelerates uniformly at 1.5 m/s$^2$.

b) if it decelerates uniformly at 1.5 m/s$^2$.

**55.** A rock rolling down a slope from rest covers a distance of 4 m in the first second. What distance will it cover in 3 s?

**56.** A car starts from rest and accelerates to 30 m/s in 6 s. What is the displacement of the car during the fourth second? (between t=3 s and t=4 s).

**57.** When the driver of a car which moves with a velocity of 90 km/h brakes, he can stop in 50 m. Calculate

a) The deceleration of the car

b) The time elapsed before he stops.

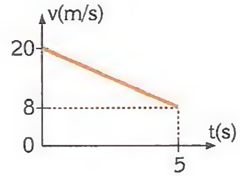
**58.** In case of an emergency, an automobile moving at a velocity of 75 km/h, stops in 6 seconds. Find the braking distance.

**59.** The braking distance of an automobile moving at a velocity of $v_1=18$ km/h is equal to $s_1=2.1$ m. What will the braking distance $s_2$ be at a velocity of $v_2=80$ km/h if it undergoes the same acceleration in both cases.

**60.** The velocity-time graph for an object is given on the right.

a) Plot its a-t graph.

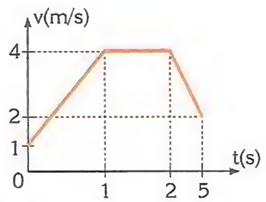
b) In which direction is the object moving?



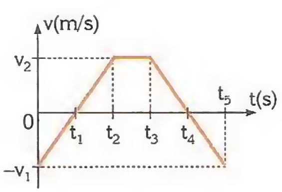

**61.** The velocity-time graph of an object is given on the right.

a) How many different accelerations does the object experience in 5 s?

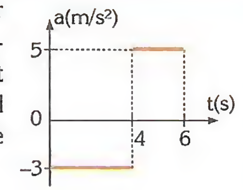
b) Plot its a-t graph.




**62.** Between which time values in the figure is the object moving in the positive direction?




**63.** The acceleration-time graph for a particle moving along the x-axis is given in the figure. What is its velocity at t=6 s if its initial velocity is 9 m/s in the positive direction?

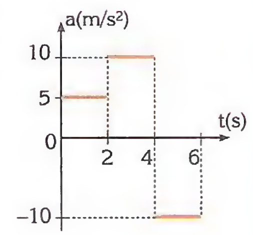



**64.** The figure on the right-hand shows the acceleration - time graph for an object moving on a straight line. Draw the velocity - time graph

a) Taking $v_0=0$,

b) Taking $v_0=8$ m/s

c) Taking $v_0=-30$ m/s.

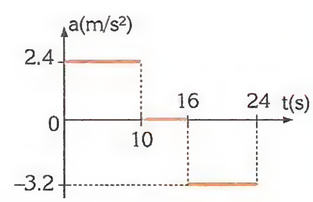



**65.** An object accelerates as shown in the graph, starting from rest. Find its velocity

a) At t=10 s

b) At t=14 s

c) At t=20 s




**66.** A machine gun projects a bullet through its barrel, of length 41.5 cm, at a muzzle velocity of 616 m/s. At what acceleration and for how many seconds is the bullet moving in the barrel?

67. An airplane can take off when it reaches a speed of 360 km/h. If the runway is 2 km long, what is the minimum value of acceleration required for the plane to be able to take off?

68. A train, accelerating uniformly, travels a distance of 240 m in 20 s and reaches a velocity of 16 m/s. What is the initial velocity and the acceleration of the train?

69. A motorcyclist and a cyclist simultaneously begin to move from rest. The acceleration of the motorcyclist is three times greater than the acceleration of the cyclist. By how many times will the velocity of the motorcyclist be greater?

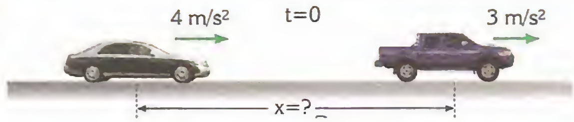
a) After the same interval of time as the cyclist.

b) After moving the same distance.

70. An automobile travels a distance of 12 km in 4 minutes. It then moves the next 12 km in 2 minutes. Calculate its acceleration, which is constant throughout the motion. (Hint: For an object moving with constant acceleration, the average velocity equals the instantaneous velocity in half the journey's time)

71. A train which can reach a maximum velocity of 90 km/h leaves a station accelerating at 1.25 m/s$^2$. When it approaches the next station, it stops by decelerating at 1.25 m/s$^2$. If the train travels the distance between two stations in 2 minutes, what is the distance between the two stations in kilometres?

72.




On a straight road a lorry and a car are at rest at a fixed distance apart. The lorry then starts moving with an acceleration of 3 m/s$^2$ and the car starts moving with an acceleration of 4 m/s$^2$. After the lorry travels 96 m, the car overtakes the lorry.

a) How many seconds later will the car overtake the lorry?

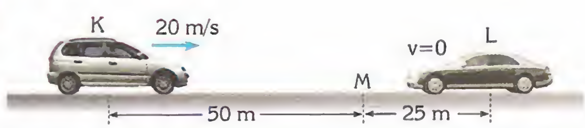
b) How many metres is the car behind the lorry at the start?

c) What are the velocities of the two vehicles when the car overtakes the lorry?

d) Draw the velocity-time, position-time, and acceleration-time graphs of each vehicle on the same coordinate axis.

(Take the initial position to be zero for both of the vehicles.)

73.




Vehicle L is at rest, and vehicle K is moving at a velocity of 20 m/s. When the distance between the vehicles is 75 m, K starts to slow down at a constant deceleration and, at the same instant, L starts to speed up at a constant acceleration towards K.

If vehicle K stops when the two vehicles meet at point M, as shown in the figure, what is the velocity of vehicle L at this point?

74.


At the instant lorry K, which is moving at a constant velocity of 20 m/s, passes point M, lorry L starts accelerating at 4 $m/s^2$ from rest in the direction shown in the figure. If the speeds of the lorries become equal at point P, how many metres is the distance $|MN|$?

75. Alexander and Veronica are two runners in a long-distance race. They are both moving at 4 m/s when Veronica is 18 m behind Alexander. When Alexander is 72 m away from the finish line, Veronica accelerates but Alexander keeps moving at 4 m/s. What minimum acceleration is required by Veronica to overtake Alexander and win the race?

### 2.6 Freely Falling Objects

76. What is the relationship between the acceleration, velocity and time for a freely falling elephant and a freely falling leaf striking the ground in an evacuated environment?.

77. Are the distances that a freely falling body travels, in equal time intervals of 1 s, equal?

78. An object falls freely with an acceleration of *9.8 $m/s^2$*. If the object were thrown downwards with an initial velocity, would its acceleration be greater than *9.8 $m/s^2$*? What would the result be if the same experiment was carried out in an evacuated environment?

Take the acceleration of free-fall as $g=10$ $m/s^2$ in the following questions.

79. If there were no air resistance, what sort of danger would await us on rainy days? If there were no air resistance, at what velocity would a rain drop strike the ground from a cloud of height 1200 m above the ground?

80. A stone is released freely from a tower of height 80 m.

a) How long does it take to reach the ground?

b) At what velocity does it strike the ground?

c) Calculate its velocity and height after 2 s.

81. An object, thrown upwards at $v_0$, reaches a maximum height of 100 m.

a) Calculate the initial velocity $v_0$.

b) Calculate the total flight time.

c) Calculate the height of the object when $v=30$ m/s.

82. An object is thrown vertically downwards from a height, at an initial velocity of 50 m/s. If it strikes the ground at 70 m/s, calculate the initial height of the object from the ground.

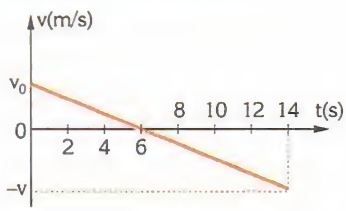
83. A stone is thrown upwards at 20 m/s from a bridge 60 m above sea level. Calculate

a) Its total time of the flight.

b) Its velocity just before it strikes the sea.

c) How many metres is the stone above sea level at $t=3$ s.

**84.**




The velocity-time graph of a stone thrown from the top of a building and striking the ground is as shown in the figure. How many metres high is the building?

**85.**

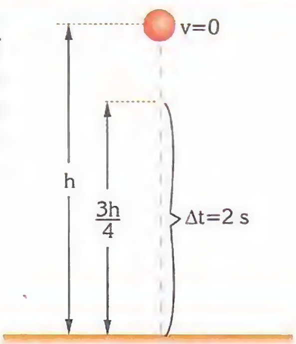
You can calculate a friend's reaction time in a simple experiment as follows: Hold a banknote (or a ruler) between your friend's fingers, just over his hand. Then ask your friend to catch it when you release it. What is the maximum reaction time possible to catch a 16 cm banknote, as shown in the figure above?

**86.** A hot air balloon is rising at a constant speed of 10 m/s, as shown in the figure. An object is released from the balloon and falls freely. If the object falls to the ground in 8 s, find

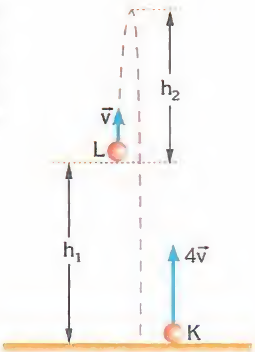

a) The height of the balloon from the ground at the moment the object is released.

b) The velocity of the object at the moment it strikes the ground.

**87.** If an object which is left to fall freely from a height of h, travels the distance 3h/4 in the last 2 s before striking the ground, from what height was it dropped?

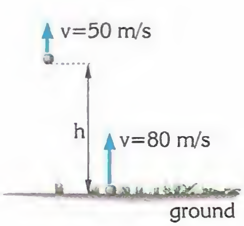



**88.** Objects K and L are thrown at the same instant of time, as shown in the figure and meet at height $h_1$. What is the ratio $\frac{h_1}{h_2}$?




**89.** Two objects are released from the same height within a time interval of 1 s. How many seconds after the second object is released will the distance between the two objects be 20 m?

**90.**




Two objects are thrown vertically upwards at the same time at speeds of 50 m/s and 80 m/s, as shown in the figure. What should the height, h, be so that both objects land at the same time?

# Motion in Two Dimensions

## CHAPTER 3

*In Chapter 2 motion in a straight-line was described. However, the discussion was confined to one dimension only. Therefore, the direction of vectors such as displacement, velocity, and acceleration was limited to the assignment of positive (+) or negative (-) signs. In this chapter, the discussion will be extended to motion in a two-dimensional plane, where objects move in both x and y directions at the same time. Later on in this chapter relative motion of an object is described , which describes the state of a body according to different observers.*

## 3.1 PROJECTILE MOTION

Consider the curved flight of an object thrown into the air. This is a special form of motion in two dimensions called **projectile motion**, and objects undergoing such motion are called projectiles. The parabolic path which is common to all objects in projectile motion is called a **trajectory**. See the examples from everyday life, as shown in Figure 3.1.

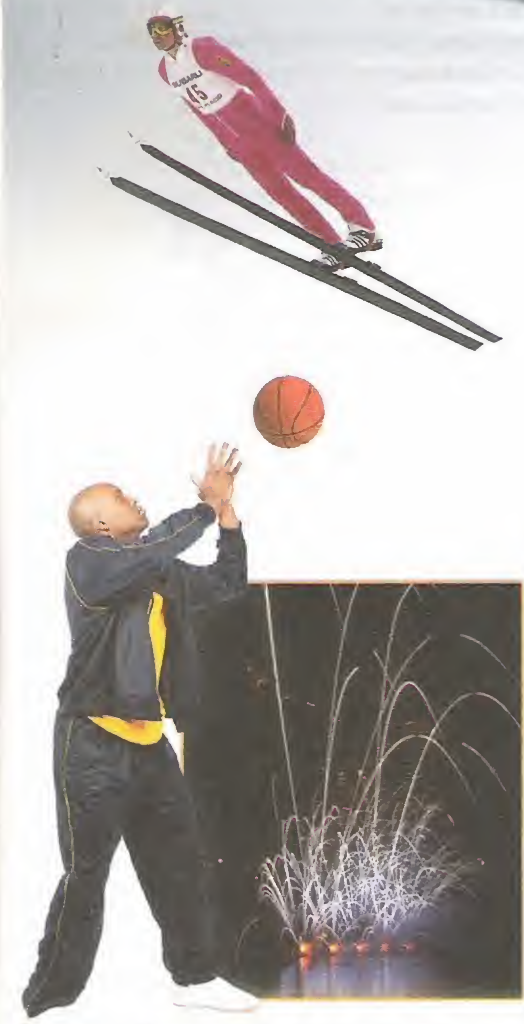


Figure 3.1 Some examples of trajectories; a basketball, fireworks and the trajectory of a ski jumper.

Let's consider the projectile motion of a ball, as shown in Figure 3.2.a. The origin of the x-y coordinate system is placed at the point from which the ball is thrown into the air.

Let's resolve the initial velocity of the ball $\vec{v}_0$ into its components along the x and y directions (Figure 3.2.b). Since the angle with the horizontal is θ;

$$v_{0x} = v_0\cos\theta \qquad v_{0y} = v_0\sin\theta$$

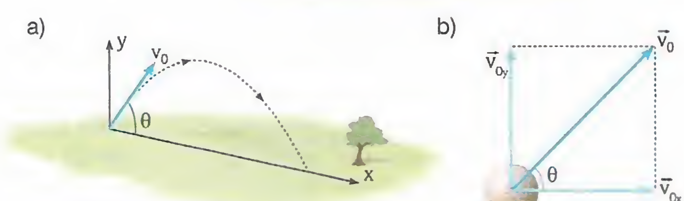



Figure 3.2 a) A ball thrown at an angle to the horizontal b) Resolving the components of the initial velocity.

The trajectory of a ball projected at an angle above the horizontal is shown in Figure 3.3.

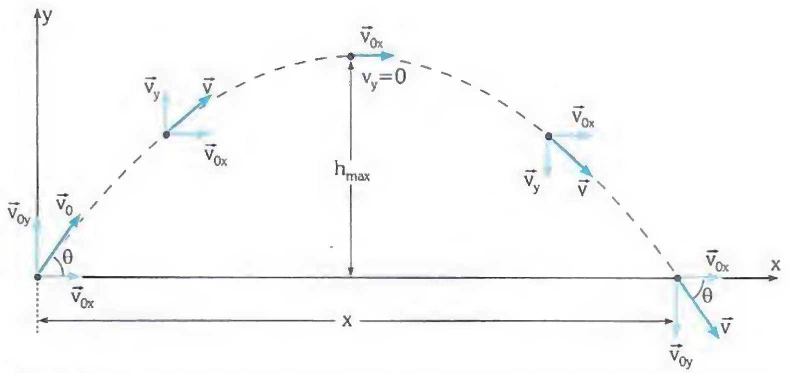



**Figure 3.3** The position and velocity of a projectile at various instants of time.

If the effect of air resistance is ignored, the projectile motion can be divided into two parts and the motion in the x and y directions can be analysed separately:

1. Motion in the x-direction: This is motion at a velocity $v_{0x}$, which remains constant during the flight. Since the motion is at constant velocity, there is no component of acceleration in this direction; ($a_x=0$). Since the velocity $v_{0x}$ is constant, the horizontal distance (also called the range) covered in any given time interval, t, is

$$x=v_{0x}t=(v_0\cos\theta)t$$

The horizontal distance covered by a projectile which returns to its original height (which is commonly ground level) is called the **range** of the projectile. The time it takes to cover its range is called the time of flight of the projectile.

2. Motion in the y-direction: This is motion at a constant acceleration which is identical to that of a freely falling object under the effect of gravity. Examine the experiments shown in Figure 3.4.a and Figure 3.4.b.

The equations derived in Chapter 2 are also valid here. If the vertical direction is assigned to be the positive (+) direction, the vertical component of initial velocity will be positive (+), and the acceleration of free fall ($g=9.8\ m/s^2$) becomes negative (–). Therefore:

$$v_y = v_{0y} - gt \qquad \Delta y = \frac{1}{2}(v_{0y}+v_y)t$$

$$\Delta y = v_{0y}t - \frac{1}{2}gt^2 \qquad v_y^2 = v_{0y}^2 - 2g\Delta y$$

The maximum vertical displacement is the maximum height ($h_{max}$) of the projectile.

Projectile motion is actually the superposition of two types of motion;

One in the horizontal - Motion with constant velocity

One in the vertical - Motion with constant acceleration

The velocity of the projectile at any instant is the resultant vector of the rectangular components $\vec{v}_x$ and $\vec{v}_y$,

$$\vec{v} = \vec{v}_x + \vec{v}_y$$

From Pythagorean theorem its magnitude is;

$$v^2=v_x^2+v_y^2$$

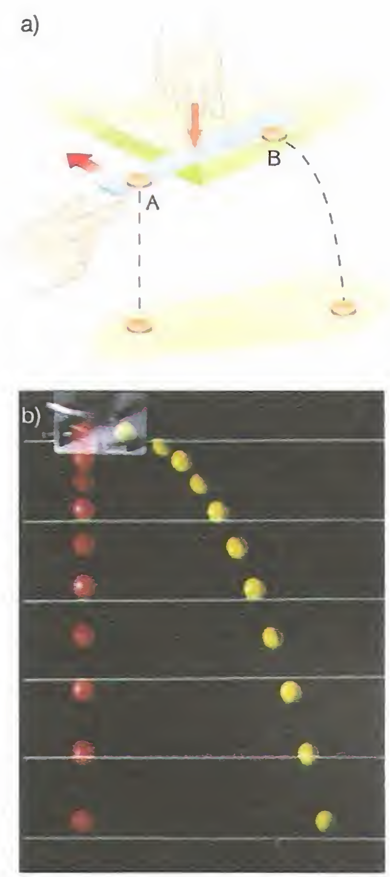



**Figure 3.4** a,b) As one coin (or ball) is dropped from rest, the other coin (or ball) is simultaneously projected sideways. Since they have the same vertical acceleration, they strike the floor at the same time.

## Example 3.1 The projectile motion of a golf ball

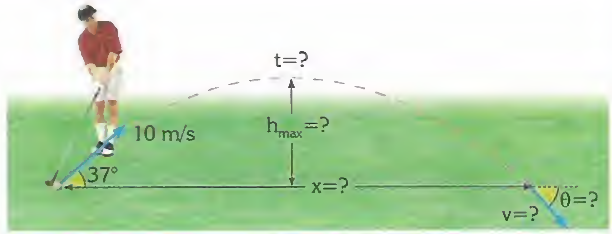
A golf ball is struck at a velocity of 10 m/s and an angle of 37° with the horizontal, as shown in the figure.

a) Calculate the time for the ball to reach its maximum height?

b) How high will the ball rise, $h_{max}$?

c) What is the range of the ball?

d) What is the ball's velocity just before it strikes the ground? (Take $g=10\ m/s^2$)




### Solution

First select the direction of the vertical component of the initial velocity to be positive. In this selection, the acceleration becomes negative because it is always directed downwards. Then calculate the components of the initial velocity as follows:

The horizontal component is

$$v_{0x} = v_0 \cos\theta = 10 \cdot 0.8 = 8\ m/s$$

The vertical component is

$$v_{0y} = v_0 \sin\theta = 10 \cdot 0.6 = 6\ m/s$$

a) The vertical component of the ball's velocity is zero at the maximum height. Using the equation

$$v = v_{0y} + gt$$
$$0 = 6 + (-10)t$$
$$t = 0.6\ s$$

b) Using the time-independent velocity equation

$$v_y^2 = v_{0y}^2 + 2g\Delta y$$
$$0 = 6^2 + 2\cdot(-10)\cdot h_{max}$$
$$h_{max} = 1.8\ m$$

c) Since the total time of flight is twice the time taken to reach the maximum height.

$$t_{flight} = 2 \cdot t_{rising}$$
$$= 2 \cdot 0.6 = 1.2\ s$$

The range of the ball is found by;

$$x = v_{0x} \cdot t$$
$$x = 8 \cdot 1.2 = 9.6\ m$$

d) The horizontal component of the ball's initial velocity is the same as $v_{0x}=8\ m/s$.

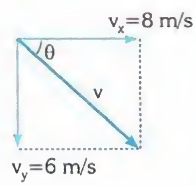
The vertical component is calculated by

$$v_y = v_{0y} + gt$$
$$v_y = 6 + (-10)(1.2)$$
$$v_y = -6\ m/s$$

We know that

$$v^2 = v_x^2 + v_y^2$$
$$= 8^2 + (-6)^2$$
$$v = 10\ m/s$$

$$\tan\theta = \frac{v_y}{v_x} = \frac{-6}{8} = -0.75$$

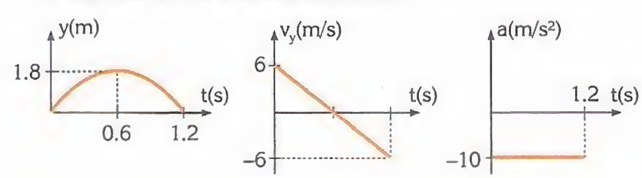



therefore;

$$\tan^{-1}(-0.75) = \theta = -37°$$

(The negative sign means that the angle is below the +x axis)

The following graphs of the motion define the behaviour of the ball in the vertical direction:



## Example 3.2

Horizontal motion

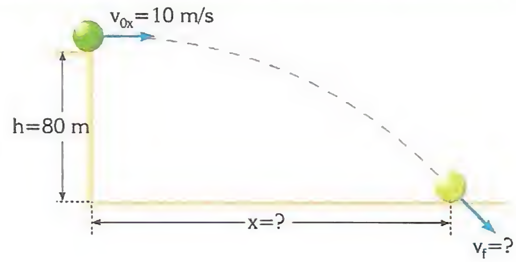
A ball is thrown at 10 m/s horizontally from the top of a building, 80 m high. Find the following

a) The time of flight of the ball.

b) The distance away from the base of the building that the ball strikes the ground.

c) The velocity of the ball just before it strikes the ground.

d) The angle the ball makes with the horizontal after a time of 1 s.




**Solution**

Since the initial velocity of the ball is horizontal, this motion is the second half of the projectile motion trajectory of an object thrown upwards from ground level.

a) We can consider the initial height of the ball as the maximum height of a projectile thrown from the ground, where the initial vertical velocity was zero. Therefore

$$\Delta y = v_{0y} \cdot t + \frac{1}{2}gt^2$$

$$h = 0 + \frac{1}{2}gt^2$$

$$80\text{ m} = \frac{1}{2}10 \cdot t^2 \quad \Rightarrow \quad t = 4\text{ s}$$

b) The range is found by

$$x = v_{0x} \cdot t$$

$$x = 10 \cdot 4 = 40\text{ m}$$

c) The vertical component of the final velocity is

$$v_y = v_{0y} + gt$$

$$= 0 + 10 \cdot 4 = 40\text{ m/s}$$

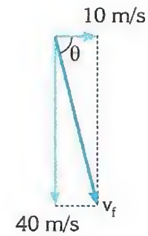
The horizontal component is constant;

$v_{0x} = v_x = 10$ m/s, therefore;

$$v_f^2 = v_x^2 + v_y^2 = 10^2 + 40^2$$

$$v_f = 41.2\text{ m/s}$$

$$\theta = \tan^{-1}\left(\frac{40}{10}\right) = 76°$$




d) After 1 s;

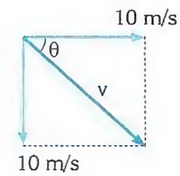
$$v_y = gt$$

$$v_y = 10 \cdot 1$$

$$v_y = 10\text{ m/s}$$

$$v_x = v_{0x} = 10\text{ m/s}$$

$$\theta = \tan^{-1}\left(\frac{10}{10}\right) = 45°$$




## Example 3.3

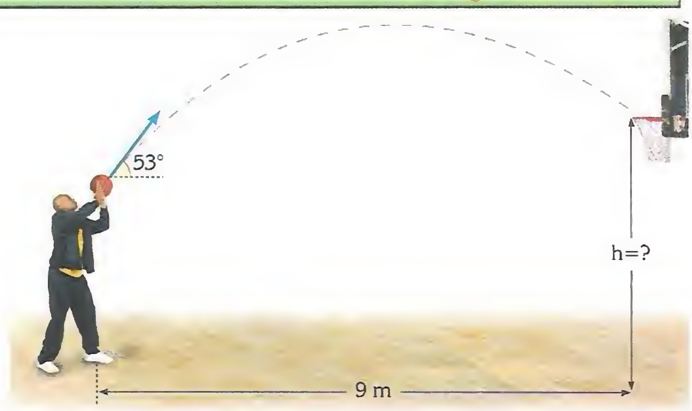
Throwing a basketball

A basketball player succeeds in passing a ball through the net when he throws it 2.3 m above the ground with a velocity of 10 m/s at an angle of 53° with the horizontal, as shown in the figure. If the player is 9 m away from the net when he throws the ball

a) How many seconds later does the ball pass through the basket?

b) What is the height of the net from the ground?

(Take $g = 10\text{ m/s}^2$ )






**Solution**

a) The time it takes the ball to just pass through the net can be found using the horizontal component of its initial velocity and the distance it travelled horizontally to the net. Applying the equation;

$$x = (v_0 \cos 53°)\, t$$

$$9 = 10 \cdot 0.6 \cdot t \quad \text{thus} \quad t = 1.5 \text{ s}$$

b) If the upward direction is chosen to be positive, $v_{0y}$ is positive and g is negative. Thus, the vertical displacement covered by the ball in 1.5 s is found using the equation

$$y_{ball} = (v_0 \sin 53°)t + \frac{1}{2}gt^2$$

$$y_{ball} = 10 \cdot 0.8 \cdot 1.5 + \frac{1}{2}(-10) \cdot (1.5)^2 \qquad y = 0.75 \text{ m}$$

Thus, the height, h, of the net from the ground is

$$h = 2.3 \text{ m} + y_{ball}$$

$$h = 2.3 \text{ m} + 0.75 \text{ m} \quad \text{thus} \quad h = 3.05 \text{ m}$$

## Example 3.4

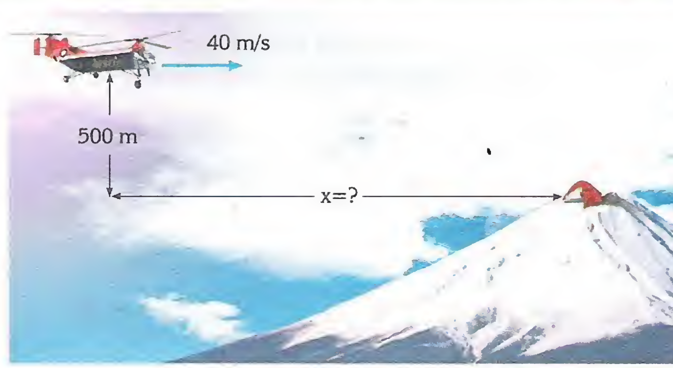
Package dropped from a helicopter

While climbing Mount Ararat, some amateur mountaineers get stranded at a given location. A rescue helicopter reduces its velocity to 40 m/s and descends 500 m to deliver an aid package to the mountaineers, as shown in the figure.

a) What horizontal distance from the location should the aeroplane release the package to land exactly where the mountaineers are trapped?

b) Calculate the velocity of the package just before it strikes the mountain.

c) What is the appearance of the path of the package according to the helicopter pilot? (Take $g = 10 \text{ m/s}^2$)




**Solution**

a) When the package is released, it is projected horizontally at the same velocity as the helicopter, with respect to the Earth. Thus, it falls 500 m freely in the vertical direction, and also it moves a distance x along the horizontal direction at a constant velocity. Selecting the downward direction to be positive, and using the equation

$$y = h = \frac{1}{2}gt^2$$

$$500 = \frac{1}{2} \cdot 10t^2 \quad \text{thus} \quad t = 10 \text{ s}$$

Thus, the distance x it covers in the horizontal direction is

$$x = v_0 t$$

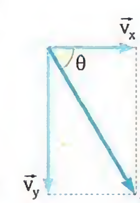
$$x = 40 \cdot 10 \quad \text{thus} \quad x = 400 \text{ m}$$

Thus, the package should be released 400 m horizontally from the mountaineers.

b) The horizontal component of the package's final velocity is the same as the initial velocity

$$v_x = v_0 = 50 \text{ m/s}$$

The vertical component of the initial velocity is zero. The vertical component of the final velocity is




$$v_y = v_{0y} + gt$$

$$v_y = 0 + 10 \cdot 10 \quad \text{thus} \quad v_y = 100 \text{ m/s}$$

Then the final velocity (velocity just before it strikes the mountain) is

$$v^2 = v_x^2 + v_y^2$$

$$v^2 = 40^2 + 100^2 \qquad \text{thus} \qquad v = 108 \text{ m/s}$$

The direction of the final velocity is

$$\tan\theta = \frac{v_y}{v_x} = \frac{100}{40} \quad \Rightarrow \quad \tan\theta = 2.5$$

$$\theta = 68.2°$$

c) According to the pilot of the helicopter, the package appears to fall freely along the vertical direction.

## 3.2 RELATIVE VELOCITY

### a. Relative Velocity in One Dimension

Imagine you are driving the car in Figure 3.5 at a constant velocity $\vec{v}$ along a straight line. On passing traffic lights, we ordinarily think of them as being stationary and ourselves in motion. Things can, however, we viewed from a different perspective if you imagine yourself as a stationary observer in the car, and observe the motion of the traffic lights. Physically it is also possible to think of the car as being at rest and the traffic lights moving at the same speed but in the opposite direction! (with a velocity of $-\vec{v}$), (Figure 3.5.a and b).

Next, think about a bus moving at the same velocity as the car, shown in Figure3.5.c. The motion of the bus with respect to the car appears to be stationary.

So the motion of the car, the bus or the traffic lights depends on the measurement of the observer. In other words, **all motion is relative to a given frame of reference.**

Let's study a numerical example: A car is moving at 30 m/s westwards and a bus passes it at 40 m/s westwards (Figure 3.6). A stationary observer standing on the side of the road would measure the velocity of the bus to be 40 m/s westwards. However, with respect to the car, the bus appears to be moving westwards at 10 m/s.

$$\text{The velocity of the bus relative to the car} = \text{the velocity of the bus} - \text{the velocity of the car}$$

$$10 \text{ m/s} = 40 \text{ m/s} - 30 \text{ m/s}$$

In general,

$$\vec{v}_{relative} = \vec{v}_{object} - \vec{v}_{observer}$$

Assume that there are two cars, A and B, moving at constant velocities of $\vec{v}_A$ and $\vec{v}_B$ with respect to an observer, who is at rest on the ground. The velocity of A relative to B, in the form of subscripts, is written as $\vec{v}_{AB}$ and calculated by,

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$$

(Note that all velocity values in relative motion problems are considered to be constant).

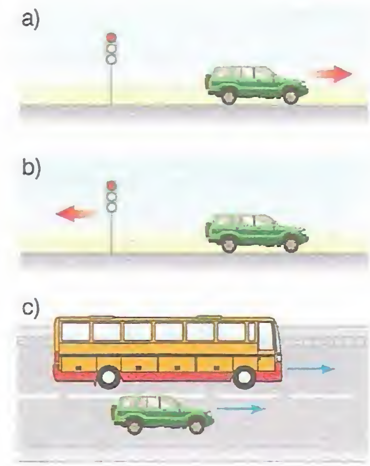



Figure 3.5 Relative motion in a given frame of reference;

a) Relative to an observer at rest on the ground (or relative to the Earth) the car is in motion, the traffic lights are stationary.

b) Relative to the driver in the car, the car is stationary, the traffic lights are moving away in the opposite direction.

c) The car and the bus moving at the same velocity appear stationary with respect to each other.

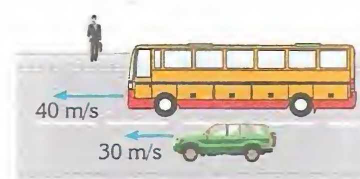



Figure 3.6 An Observer on the ground and in the car will observe the bus to be at different velocities.

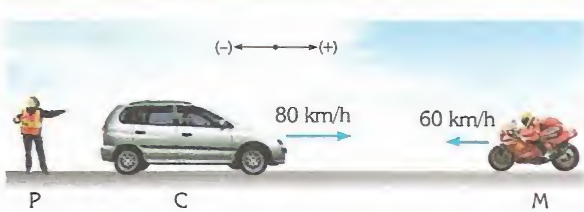
### Example 3.5 — Relative motion in one dimension

The velocity vectors of a car and a motorcycle relative to the Earth are shown in the figure. Find the velocities of the car and the motorcycle

a) A policeman, who is at rest next to the street

b) An observer in the car

c) A motorcyclist.



### Solution

All motions is in one dimension along a straight line. Velocities to the right can be assigned as positive (+) and velocities to the left can be assigned as negative (–).

a) The policeman is stationary relative to the Earth. Therefore the velocities given relative to the Earth are also the velocities relative to the policeman. thus, using the subscript terminology described on page 7, where c, p, m stand for the car, man and motorcyclist, respectively,

$$\vec{v}_C = \vec{v}_{CP} = 80 \text{ km/h}$$

$$\vec{v}_M = \vec{v}_{MP} = -60 \text{ km/h}$$

b) Relative to an observer in the car;

i. The velocity of the car is zero.

ii. The velocity of the policeman is

$$\vec{v}_{relative} = \vec{v}_{object} - \vec{v}_{observer}$$

$$\vec{v}_{PC} = \vec{v}_P - \vec{v}_C$$

$$= 0 - 80 \text{ km/h}$$

$$v_{PC} = -80 \text{ km/h}$$

(the negative sign indicates that the velocity is in the –x direction)

iii. The velocity of the motorcycle is

$$\vec{v}_{relative} = \vec{v}_{object} - \vec{v}_{observer}$$

$$\vec{v}_{MC} = \vec{v}_M - \vec{v}_C$$

$$= -60 \text{ km/h} - 80 \text{ km/h}$$

$$\vec{v}_{MA} = -140 \text{ km/h}$$

c) Relative to the motorcyclist;

i. The velocity of the car is

$$\vec{v}_{relative} = \vec{v}_{object} - \vec{v}_{observer}$$

$$\vec{v}_{CM} = \vec{v}_C - \vec{v}_M$$

$$= 80 \text{ km/h} - (-60 \text{ km/h})$$

$$\vec{v}_{CM} = 140 \text{ km/h}$$

ii. The velocity of the policeman is

$$\vec{v}_{relative} = \vec{v}_{object} - \vec{v}_{observer}$$

$$\vec{v}_{PM} = \vec{v}_P - \vec{v}_M$$

$$\vec{v}_{PM} = 0-(-60 \text{ km/h})$$

$$\vec{v}_{PM} = 60 \text{ km/h}$$

iii. The velocity of the motorcycle is zero.

Note that the relationship between the relative velocities of any two bodies A and B is

$$v_{AB} = -v_{BA}$$

such as the velocities $v_{CM} = -v_{MC}$ or $v_{PC} = -v_{CP}$ calculated above.

same order

$$v_{AB} = v_{AC} + v_{CB}$$

cancel

## Problem Solving Strategy

The order of the subscripts on the relative velocity $v_{AB}$ is expressed as the "velocity of A relative to B". Similarly, the velocity of A relative to a third frame of reference, C, is $v_{AC}$ and the velocity of the frame of reference C relative to that of B is $v_{CB}$. These are related by the simple equation shown in the right-hand figure.

Note that the two outer subscripts on the right-hand side must be in the same order as the two subscripts on the left-hand side. The two inner subscripts on the right-hand side must be the same, so they cancel.

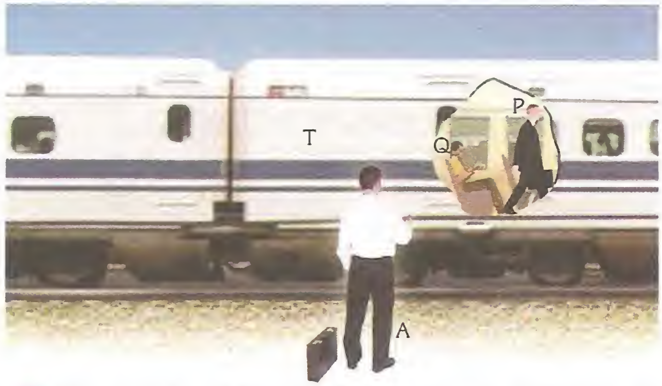
## Example 3.6

Relative motion in one dimension

A train is moving at a constant velocity of 6 m/s. A passenger in the train is also moving at a velocity of 1 m/s relative to the train, in the same direction as the train.

a) What is the velocity of the passenger relative to another passenger, Q sitting in the train?

b) What is the velocity of the passenger relative to the observer, A beside the train?




**Solution**

a) The train is stationary relative to the passengers.

$$v_{TQ} = v_{TP} = 0.$$

The velocity of passenger P relative to another passenger Q, who is at rest, $v_{PQ}$, can be written as

$$v_{PQ} = v_{PT} + v_{TQ}$$

$$v_{PQ} = (1 \text{ m/s}) + 0$$

$$v_{PQ} = 1 \text{ m/s}$$

b) The velocity of the moving passenger relative to the observer beside the train can be written as

$$v_{PA} = v_{PT} + v_{TA}$$

$$v_{PA} = 1 \text{ m/s} + 6 \text{ m/s} = 7 \text{ m/s}$$

### b. Relative Velocity in Two Dimensions

The examples solved above are some applications of relative velocity only in one dimension. The objects in these examples are moving either in the same or in opposite directions relative to each other. However, straight-line motion is rare in real life. Thus, another method must be used for analysing relative motion problems in real life, for example, for objects moving perpendicular to each other, as shown in Figure 3.7. Knowing that velocity is a vector, vector addition and subtraction can be used to solve such problems in two-dimensions.

Figure 3.7 Relative motion in two dimensions

## Example 3.7

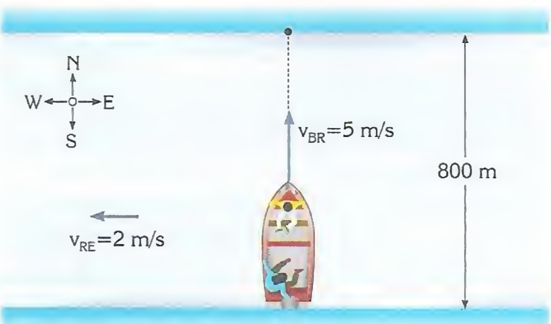
The relative motion of a boat in a river

A small boat crosses a wide river with a steady speed of 5 m/s due north relative to the water, as shown in the figure. The river also has a steady speed of 2 m/s due west relative to the Earth.

a) What is the velocity of the boat relative to an observer at rest on the ground?

b) How long does it take for the boat to cross the river, which is 800 m wide?

c) At which point does the boat intercept the other side of the river?

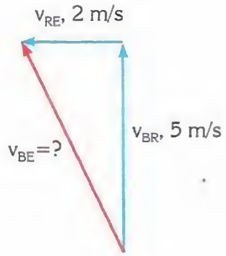


## Solution

a) Denoting the velocity of the boat relative to the river as $v_{BR}$ and the velocity of the river relative to the Earth as $v_{RE}$.

The following equation an be written in subscript notation; $v_{BE}=v_{BR}+v_{RE}$

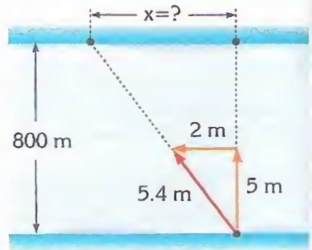
This is shown in the vector diagram as follows




Using the Pythagorean Theorem,

$$(v_{BE})^2=(v_{BR})^2+(v_{RE})^2$$
$$=(5\text{ m/s})^2+(2\text{ m/s})^2$$
$$v_{BE}=\sqrt{29}\text{ m/s}=5.4\text{ m/s}$$

b) The distances moved by the river and the boat in one second are illustrated in the diagram below. The resultant displacement of the boat relative to the Earth in 1 s is 5.4 m in the given direction.

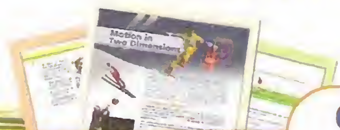



If it takes 1 s to travel 5 m horizontally, then it takes 160 s to travel 800 m which is the whole width of the river

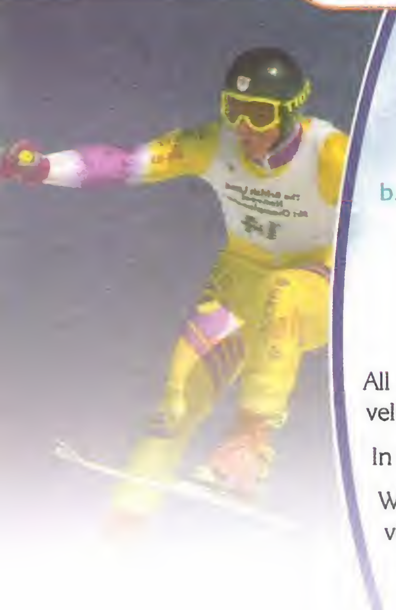
$$t=\frac{800\text{ m}}{5\text{ m/s}}=160\text{ s}$$

c) It takes 1 s to undergo a horizontal displacement of 2 m, thus in 160 s the boat covers

$x=(2\text{ m/s})\cdot 160\text{ s}=320\text{ m}$ in the horizontal direction until it reaches the other side of the river.



# Summary



Projectile motion can be analysed in the x and y directions separately:

a. In the x-direction; motion in the horizontal has a constant velocity of $v_{0x}=v_0\cos\theta$.

The range of a projectile is the horizontal distance covered by the projectile if it returns to its original height. It is given by $x=v_{0x}t=(v_0\cos\theta)t$

b. In the y-direction; motion takes place with a constant acceleration of $g=9.8\text{ m/s}^2$. The equations of motion in the vertical direction are

$$v_y=v_{0y}-gt \qquad \Delta y=\frac{1}{2}(v_{0y}+v_y)t$$
$$\Delta y=v_{0y}t-\frac{1}{2}gt^2 \qquad v_y^2=v_{0y}^2-2g\Delta y$$

All motion is considered relative to a chosen frame of reference (or state of motion of an object). The velocity of an object A relative to an object B is $v_{AB}$ and it is calculated by $v_{AB}=v_A-v_B$.

In general $v_{relative}=v_{object}-v_{observer}$

When there are more than two reference frames, a double subscript rule can be applied to relative velocities as follows:

$$v_{AB}=v_{AC}+v_{CB}$$

where the two outer subscripts on the right-hand side are in the same order as the two subscripts on the left-hand side; their inner subscripts, C, are the same and thus they cancel each other out.

In the case of objects moving in two dimensions, the principles explained above are still valid except that the relative velocities are found using vector addition.

# QUESTIONS AND PROBLEMS

### 3.1 Projectile Motion

Ignore air resistance, take $g=10\ m/s^2$

**1.**


An object is thrown at an angle of θ to the horizontal, as shown in the figure. How do the acceleration and the horizontal and vertical velocities of the object change? When do the magnitudes of these quantities reach their maximum and minimum values?

**2.** A player kicks a football at a velocity of 50 m/s, making an angle of 37° with the horizontal, as shown in the figure.


a) Calculate the vertical and horizontal components of its initial velocity, ($v_{0x}=?$, $v_{0y}=?$)

b) How long does the football take to reach its maximum height?

c) For how long is the ball in motion? (time of flight, $t_f=?$)

d) What is the maximum height that the ball can reach?

e) What is the horizontal range of the ball?

f) What are the height and velocity of the ball 2 s later?

g) What are the height and velocity of the ball 4 s later?

**3.** A signal flare is fired with a muzzle velocity of 180 m/s at an angle of 70° above the horizontal for someone to notice. What is its time of in flight? (It moves until it drops to the same horizontal level from which it was thrown).

**4.** How would the time of flight, the range and the maximum height of an object exhibiting projectile motion change if

a) only the vertical component of its initial velocity is increased?

b) only the horizontal component of its initial velocity is increased?

**5.** The girl in the picture is performing a long jump which is an example of projectile motion. Bob Beamon, an American athlete, set a long-standing world record for the long jump in 1968 with a jump of 8.90 m. His world record stood for 23 years, and was named one of the five greatest sports moments of the 20th century. If Beamon left the ground at an angle of 32.0° to the horizontal and at a velocity of 9.5 m/s in his long jump, how long did he stay in the air?

(Take sin 32°=0.53, cos 32°=0.85)

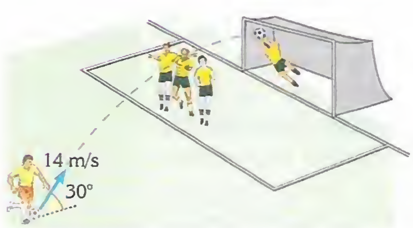
6.


A ball is thrown from the surface of the Moon at a velocity of 25 m/s making an angle of 37° with the horizontal, as shown in the figure. Take the acceleration of free-fall on the Moon to be 1.64 $m/s^2$.

a) How many seconds later will the ball strike the ground?

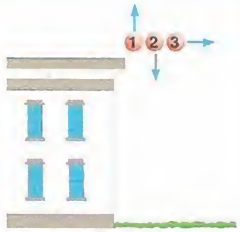
b) Find the range of the ball.

7.




In a football match, a player kicks the ball at a velocity of 14 m/s, making an angle of 30° with the horizontal, as shown in the figure. The ball passes over the defence players and the goal keeper catches it 2 m above the ground. What is the horizontal distance covered by the ball during its flight?

8. Three balls are sequentially thrown at the same instant from the top of a building, at the same speed. Ball 1 is thrown vertically upwards, ball 2 is thrown vertically downwards and ball 3 is thrown horizontally, as shown in the figure.



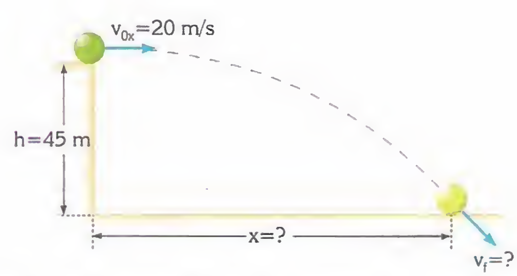

a) Do all balls reach the ground at the same time? If not, in what order do they fall?

b) Do they strike the ground at the same speed? If not, list their velocities in order of magnitude.

c) Do the masses of the balls affect the times they take to fall?

9. An object is thrown horizontally from a height. How does the time of flight and the range of the object change, when the magnitude of the initial velocity is tripled?

10.




A stone is thrown at 20 m/s horizontally from the edge of a cliff 45 m high, as shown in the figure.

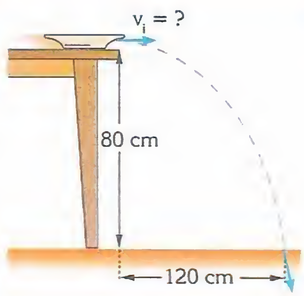
a) Calculate the time it takes to complete its trajectory. ($t_f$=?)

b) Calculate the horizontal distance covered by the stone during the flight. (x=?)

c) Calculate the velocity of the stone just before it strikes the ground.($v_{final}$=?)

d) Calculate the angle the stone makes with the horizontal after 2 s.

11. A plate is pushed horizontally over a 80 cm high table, as shown in the figure. If it lands at a point 120 cm away from the table

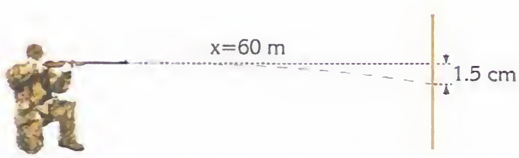



a) Find the time the plate takes to land on the floor.

b) What is the initial velocity of the plate?

c) At what velocity does the plate strike the floor?

12.

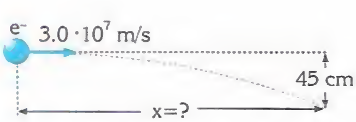



A rifle is aimed horizontally at a point 60 m away, as shown in the figure. The bullet misses the target and strikes the wall 1.5 cm below the target point. If the effects of air resistance are neglected

a) Calculate the time that the bullet remained in the air.

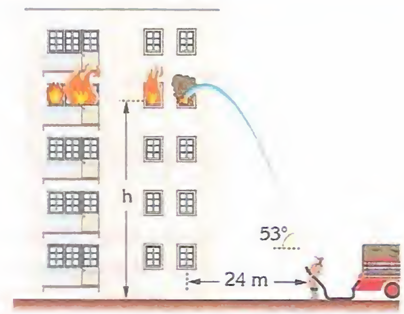
b) Find the bullet's muzzle velocity.

13.




An electron is launched at a horizontal velocity of $3 \cdot 10^7$ m/s, as shown in the figure. How many metres should the electron travel in the horizontal direction in order to have a declination of 45 cm?

14.



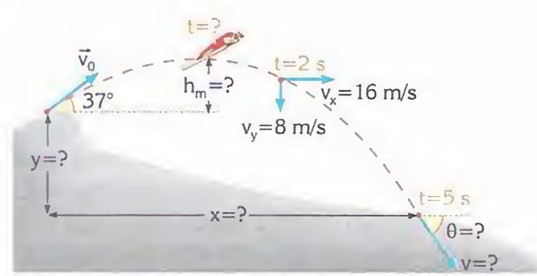

To extinguish a fire on the fourth floor of a building, a fireman squirts water into the window of the fourth floor, from a point 24 m away from the building, as shown in the figure. The water rushes out of the hose pipe at a velocity of 20 m/s, making an angle of 53° with the horizontal. If the fireman holds the pipe 1 m above the ground;

a) How many seconds does it take for the water to reach the window?

b) What is the height of the fourth floor window from the ground?

c) What is the velocity of the water at the moment it enters the window?

15.



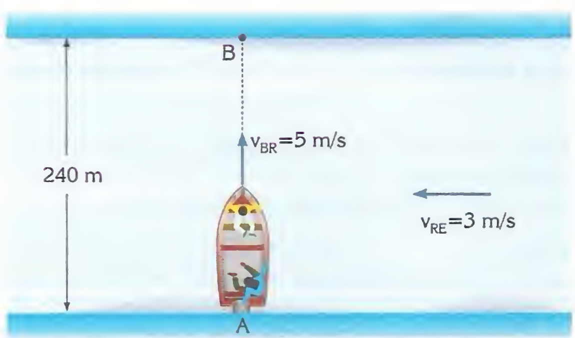

A skier leaves a ramp with a velocity $\vec{v}_0$ at an angle of 37° with the horizontal and 5 s later he lands on the ground, as shown in the figure.

If his velocity components, 2 s after he leaves the ramp, are as shown in the figure, find

a) the initial velocity $\vec{v}_0$ of the skier at the moment of leaving the ramp.

41. The pilot of a helicopter, which flies due north at a velocity of $3\vec{v}$ relative to the Earth, observes an aeroplane flying at a velocity of $4\vec{v}$ due east. What is the velocity of the plane relative to the ground?

42.




A small boat which has a velocity of 5 m/s on still water carries passengers from one side of a river to the other side, as shown in the figure. The river is 240 m wide and flows at a constant velocity of 3 m/s.

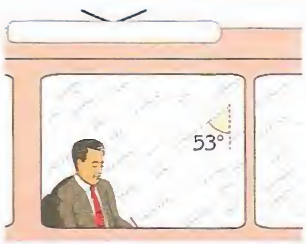
a) If the boat starts moving at a constant velocity from point A, perpendicular to the direction of the water flow, what is its velocity relative to an observer on the ground?

b) How long does it take for the boat to reach the other side of the river? How far from point B does the boat drift?

c) If the boat needs to reach point B on the other side of the river, in which direction should it head?

d) How long does the boat take to reach point B?

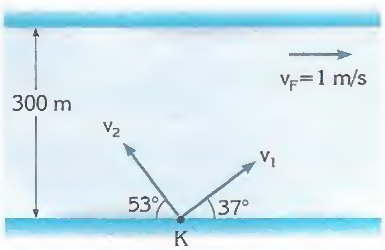
43. On a rainy day when there is no wind, rain drops fall vertically towards the ground. On such a day, while a train moves at a constant velocity of 40 km/h, an observer looking out of the train's window observes the falling rain drops making an angle of 53° with the vertical, as shown in the figure. What is the magnitude of the velocity of the rain drops




a) Relative to the train?

b) Relative to Earth?

44.



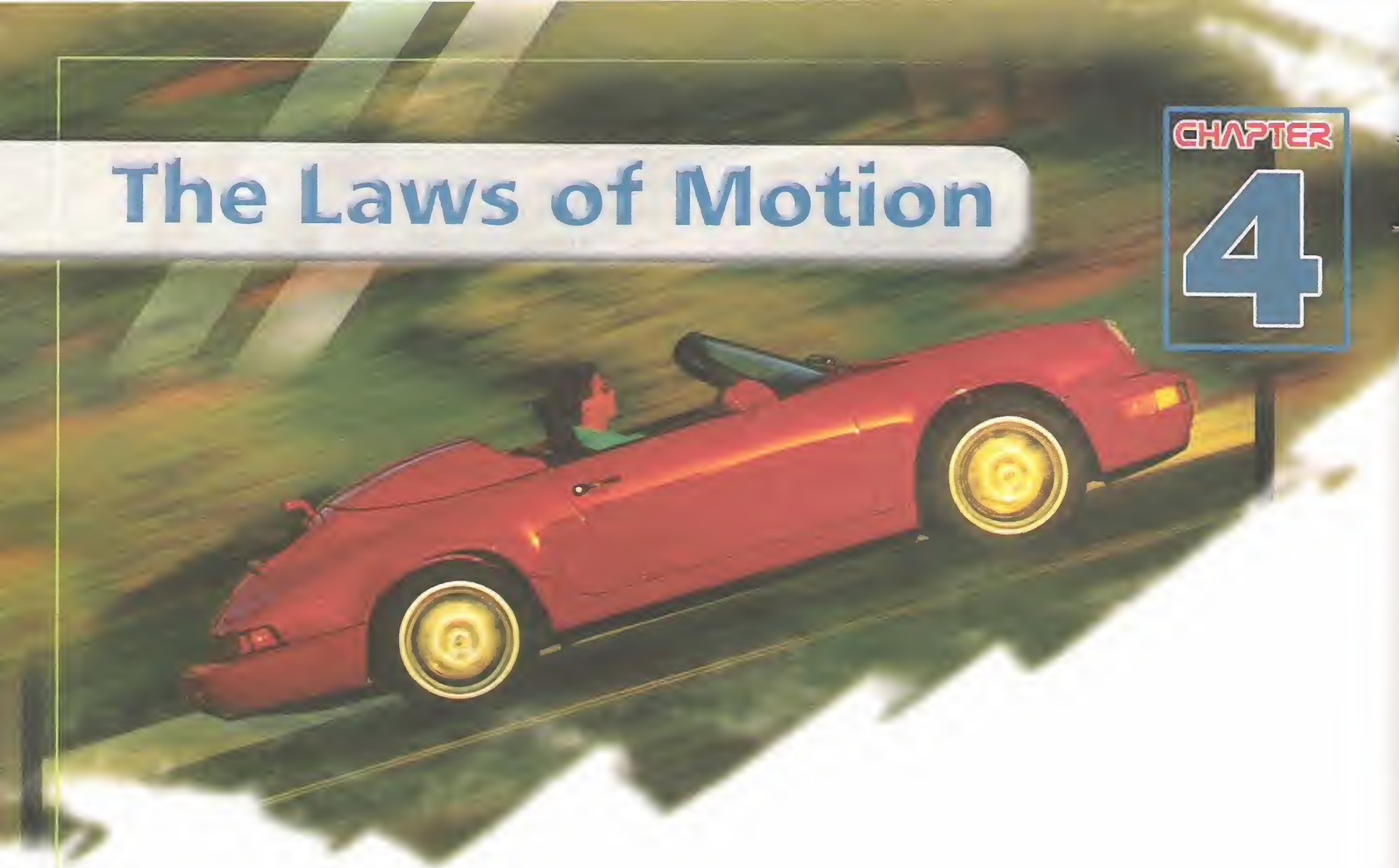

In a 300 m wide river, flowing with a velocity of 1 m/s, two boats, one with a velocity of $v_1$ = 20 m/s relative to the Earth and the other with a velocity of $v_2$ = 15 m/s relative to the river, start moving from point K, as shown in the figure. What will the distance between the boats be when they reach the other side of the river?

45. Two trains are moving towards each other at velocities of 36 km/h and 54 km/h. A passenger in the first train observes that the second train passes his train in a time interval of 6 seconds. What is the length of the second train?

46. In a given escalator it takes 30 s to move up to the next floor. When the escalator is out of service, a person walking up the stairs at a constant velocity takes 45 s to climb up to the next floor. How long will it take if he walked at the same constant velocity on a moving escalator?

CHAPTER 4

# The Laws of Motion



*In Chapter 2 The motion of objects was described with the quantities of kinematics such as displacement, velocity, and acceleration. However, these quantities cannot describe all aspects of motion, for example what starts objects moving and what causes change in their motion. In this chapter the causes and factors affecting motion will be discussed, using the concept of force and mass. The three fundamental laws of motion, discovered and formulated by Isaac Newton will be discussed. These laws explain rest, constant motion, accelerated motion, and they also describe how balanced and unbalanced forces behave; and thus give rise to various states of motion. The branch of physics dealing with the causes of motion is called dynamics.*

## 4.1 THE CONCEPT OF FORCE

Force is any kind of push or pull on an object. When a car is pushed, as shown in Figure 4.1.a, a force is exerted upon it. When a locomotive pulls a train, as shown in Figure 4.1.b, it exerts a force on the train. When a ball is kicked, as shown in Figure 4.1.c, a force is exerted upon it. When a hammer strikes a nail, or a motor pulls a lift, or the wind blows the leaves of a tree, a force is being exerted. Such a force between two objects which are in contact is called a **contact force**.

There are also forces, called **field forces**, which act even when objects are separated by a distance. Gravitational force, magnetic force and electric force are known as field forces. When an object is released near the surface of the Earth, it falls down towards the surface of the Earth due to the gravitational force (also called the weight of the object) between the object and the Earth, which are not originally in contact, as shown in Figure 4.1.d. Two magnets repel each other when their same poles are brought closer or attract each other when their opposite poles are brought closer, even though the magnets are not in contact, as shown in Figure 4.1.e. The attractive electric force between two like charges and the repulsive electric force between two unlike charges, as shown in Figure 4.1.f is an example of a field force.

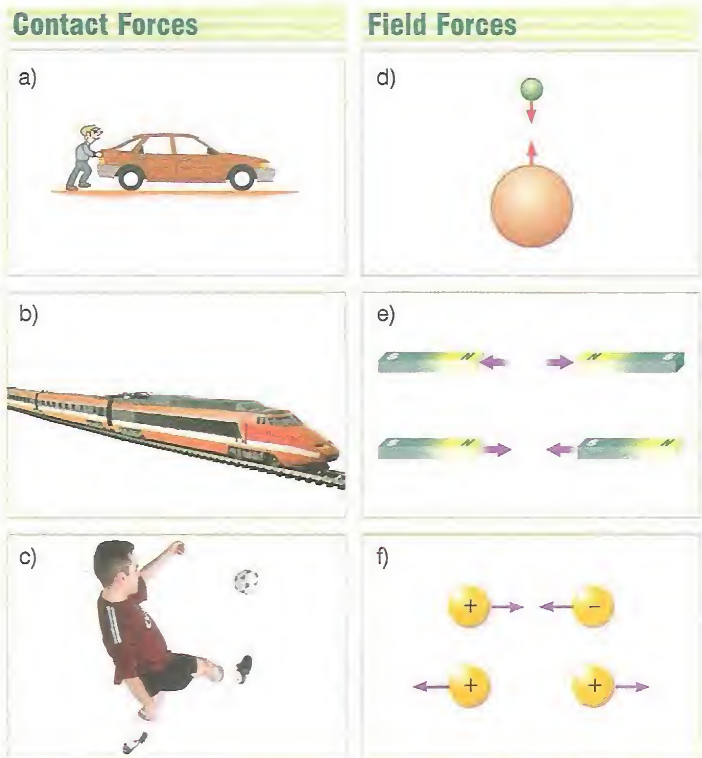



**Figure 4.1** Some examples of contact and field forces

The forces applied on the gymnast in Figure 4.2 by the strings, are contact forces, whereas the gravitational force applied upon him by the Earth (his weight) is a field force.

The magnitude (or strength) of a force can be measured by a spring mechanism called a dynamometer (or a spring balance). In a dynamometer, a spring is fixed to one end of the mechanism. When a force is applied to the other end of the spring, the spring extends in proportion to the applied force. The greater the force, the greater the extension. The dynamometer scale is calibrated using known forces (the weights of standard masses), as shown in Figure 4.3.

Since force is a vector quantity, it has a direction as well as a magnitude and is represented by an arrow whose length is proportional to the magnitude of the force. The SI unit of force is the Newton and is abbreviated by N.

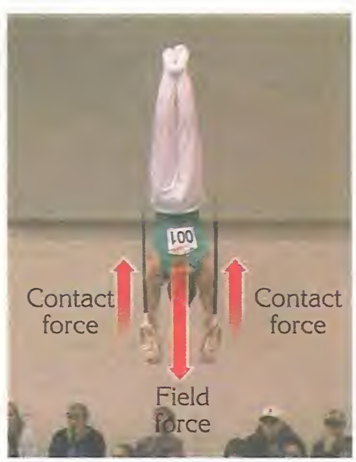



**Figure 4.2** The forces applied upon the gymnast by the ropes are contact forces and the gravitational force exerted upon him by the Earth is a field force.

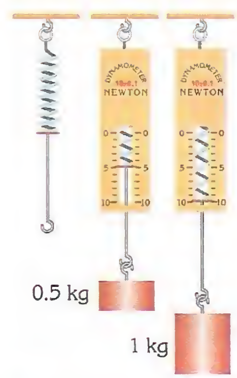



**Figure 4.3** The scale of a dynamometer is calibrated by means of the weights of standard masses.

## 4.2 NET FORCE

If more than one force acts simultaneously on the same object, it is the **net force** that determines the motion of the object. The net force, $\vec{F}_{net}$, is the vector sum of all forces acting on the object. It is also called the **resultant force**, or the **unbalanced force**. Forces are added by the rules of vector addition, as shown in Figure 4.4.

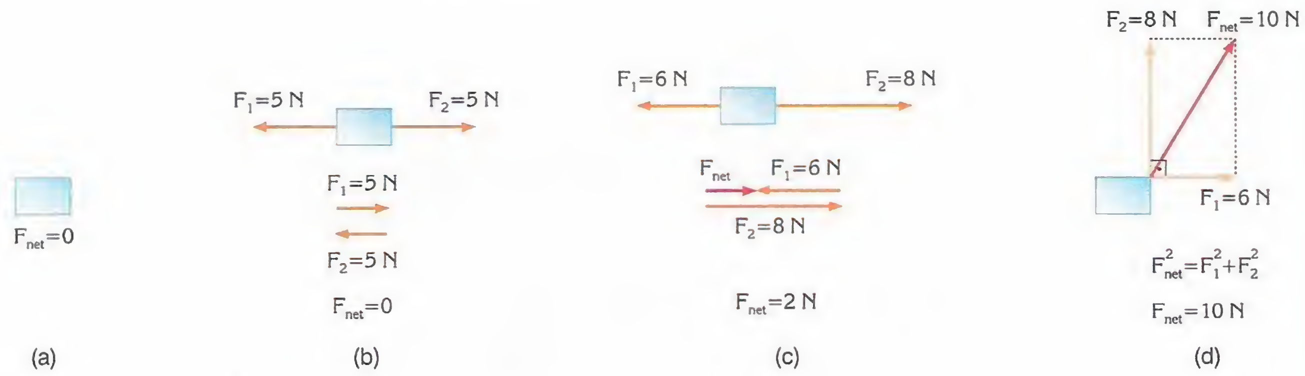
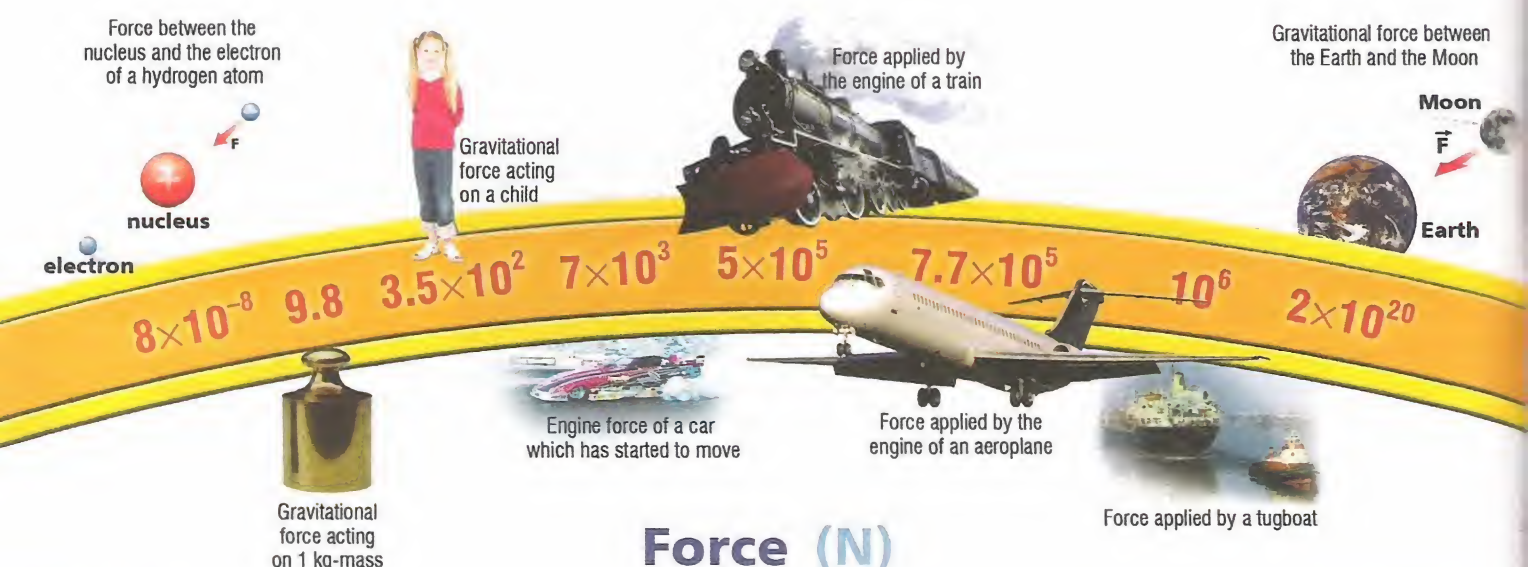



**Figure 4.4** The net force is the vector sum of the forces acting on an object

Note: In the applications of the net force in the problems solved in this book, instead of $F_{net}$ sometimes the symbol $\Sigma F$ will be used, which represents the sum (resultant) of forces acting on an object or a system.




## 4.3 THE FIRST LAW OF MOTION

Assume that an object is at rest in space, where no force acts upon it, as shown in Figure 4.5. Unless a net force (unbalanced force) acts on it, it remains at rest. If the object initially moves with a constant velocity, it continues to move with the same constant velocity unless it experiences a net force.

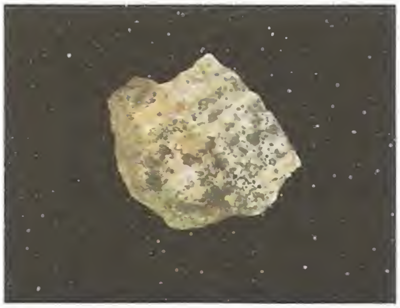


Figure 4.5 An object at rest in space where no force of gravity acts. Unless a net force acts on it, it remains at rest. If the object initially were moving with a constant velocity, it would continue to move with the same constant velocity until it experiences a net force.

From this simple experiment, the first law of motion can be inferred, it was stated by Newton more than 3 centuries ago as follows:

If the net force acting on an object is zero

- If it is at rest, it will stay at rest.
- If it is moving, it keeps on moving at a constant velocity (a constant speed in a straight line)

### a. Inertia

**Inertia is the tendency of an object to resist any change in its state of motion.**

If an object is at rest, it tends to remain at rest. If the object is in motion with a constant velocity, it tends to continue to move with the same velocity, ( the same magnitude and direction)

**The first law** is often called **the law of inertia** because it states that in the absence of a net force, a body will preserve its state of motion.

The following situations can be discussed in order to understand the law of inertia. If you are in a car which is at rest.

- As the car starts moving with an acceleration in a straight line, as shown in Figure 4.6, you can feel the car seat exerting a force on your back which acts to push you forward. Since your body resists the change in its resting state, you experience and feel this force. This force overcomes your inertia and puts you in motion with the same velocity as the car.
- Let's assume that your car is taking a turn to the right, as shown in Figure 4.7. Your body resists the change in the direction of its velocity due to your inertia and tries to keep moving on the straight line. And you feel as though you are being pushed to the left.
- Finally if the car breaks suddenly, as shown in Figure 4.8, since your body tends to move with the velocity that your car had before breaking, because of your inertia it reacts to this change in its velocity by moving forward.

Figure 4.6 As the car begins accelerating, the body resists the acceleration and tends to remain at rest due to inertia.

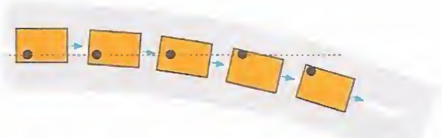


Figure 4.7 As the car takes a turn to the right, a body inside it resists the change in the direction of its velocity due to its inertia and tries to keep moving on the straight line

Figure 4.8 As the car slows down, a body inside tends to move with the velocity that the car had before breaking, due to its inertia.

Here are some other examples of inertia;

- When a hard surface is struck with the back of a hammer, it stops suddenly and the hammer head feels tightened.
- When a table cloth, on which there are some plates, is pulled rapidly from under the plates, the plates remain on the table.
- When a sheet of kitchen paper is pulled slowly, more and more paper rolls off. However, when a sheet is pulled quickly, it is torn off the roll since the pulling force doesn't have  enough time to overcome inertia.

## b. Mass is a Measure of Inertia

Mass is a measure of the response of an object to an external force. The greater the mass of an object, the greater the inertia and the less that object accelerates (changes its state of motion) under the action of a net force. Consider that if a net force causes a 1-kg object to gain an acceleration of 6 $m/s^2$, when applied to a 2-kg object, the same net force produces an acceleration of 3 $m/s^2$. From this example a relationship can be set-up between the accelerations and the masses of the objects experiencing the same net force.

$m_1 = 1$ kg $\qquad a_1 = 6\ m/s^2$

$m_2 = 2$ kg $\qquad a_2 = 3\ m/s^2$

So the relationship between the masses and their accelerations is

$$\frac{m_1}{m_2} = \frac{a_2}{a_1}$$

If one of the masses in this equation is known, the unknown mass of an object can be found after the accelerations are measured.

Finally, mass is a scalar quantity, it has no direction. It is always positive. It obeys the rules of ordinary arithmetic as other scalar quantities do.

# 4.4 THE SECOND LAW OF MOTION

The first law of motion describes what happens when the net force acting on an object is zero.

In that case, the object either remains at rest or moves in a straight line with a constant speed.

The second law of motion explains what happens when the net force acting on an object is not zero.

As stated in the simple experiment of the previous section, a net force acting on an object causes it to accelerate.

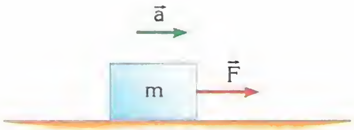



**Figure 4.9** Force F causes the object of mass m to accelerate at a value of $\vec{a}$.

Now assume that an object of mass m is pulled along a frictionless horizontal surface, exerting a horizontal force, $\vec{F}$, on it as shown in Figure 4.9. In this case, the net force acting on the object is $\vec{F}$ and the object gains an acceleration $\vec{a}$. If the force is increased to $2\vec{F}$, the acceleration increases to $2\vec{a}$. If the applied force is tripled, the acceleration triples and so on. From such experiments it can be concluded that the acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass.

From all these experiments, **the second law of motion** is stated as follows

**The acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass.**

This law was discovered and formulated by Newton as

$$\vec{F}_{net} = m\vec{a}$$

Note that net force and acceleration always have the same direction.

The unit of force, Newton, was derived from the equation of the second law of motion in terms of the fundamental units of mass, length, and time

$$1\ \text{N} = 1\ \text{kg} \cdot 1\ \text{m/s}^2$$

where 1 Newton can be defined as the force which produces an acceleration of 1 $\text{m/s}^2$, when applied on a 1-kg mass.

## a. Weight

We are well aware that objects are attracted to the Earth. The force exerted by the Earth on an object is called the weight of that object. This force is directed towards the centre of the Earth and its magnitude changes with location. At sea level the Earth exerts a force of 9.8 N on a mass of 1 kg. Experiments show that in the same location the ratio of the weight of any object to its mass gives the same value, called the gravitational field strength, g.

$$\frac{w}{m} = g \quad \text{(gravitational field strength of unit N/kg)}$$

From the equation above a relationship can be found between weight and mass

$$w = mg$$

Here it should be noted that weight is a vector quantity and depends on $\vec{g}$, which varies with geographic location, whereas mass, which is a scalar quantity, is the same everywhere in the universe.

### We can obtain the mass of an object in two different ways

1. The object can be given an acceleration, a, by exerting a net force, $\vec{F}_{net}$ on it and these quantities measured. Then the ratio of the net force to the acceleration, from the second law of motion, gives the mass of the object, which is called inertial mass.

$$\frac{\vec{F}_{net}}{\vec{a}} = m \qquad \text{Inertial mass}$$

2. The weight, $\vec{w}$ of the object can be measured with an equal arm balance. Dividing the weight by the gravitational acceleration, $\vec{g}$, gives the mass of the object, which is called gravitational mass

$$\frac{\vec{w}}{\vec{g}} = m \quad \text{Gravitational mass}$$

Gravitational mass and inertial mass have been measured in these two different ways. They were observed to be the same.

An object falls freely due to the gravitational force (that is, weight $\vec{w}=m\vec{g}$) acting upon it.

From the equation of the second law of motion

$$\vec{a} = \frac{\vec{F}}{m} = \frac{\vec{w}}{m} = \frac{m_{grav}\vec{g}}{m_{inertia}} = \vec{g}$$

where $m_{grav}$, is the gravitational mass and $m_{inertia}$ is inertial mass.

Here it is observed that the acceleration of a freely falling body and gravitational field strength are equal. Due to this, gravitational field strength is also called gravitational acceleration.

## b. Inertial Reference Frames

An inertial reference frame is a coordinate system in which the first law of motion is valid: in this coordinate system, an object at rest remains at rest and an object in motion with a constant velocity continues to move with the same velocity, unless an unbalanced force (net force) acts on it. An inertial reference frame is a coordinate system which does not accelerate.

An inertial reference frame can be explained by considering the following situation. Imagine that you are standing still on roller skates on a bus.

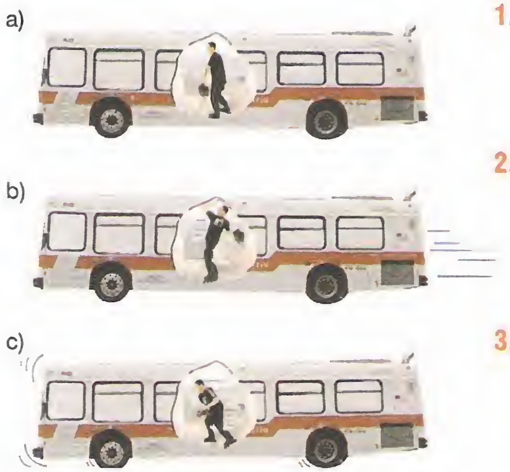



Figure 4.10 a) The bus at rest is accepted to be an inertial reference frame. b) The bus moving with constant velocity is also accepted to be an inertial reference frame. c) The accelerating bus cannot be accepted to be an inertial reference frame because the first law of motion is not valid in it.

1. You and the bus are at rest, as shown in Figure 4.10.a. Obviously your acceleration and the net force acting upon you is zero, relative to the bus. Thus, the first law of motion is valid in the bus. Thus, the bus at rest can be accepted to be an inertial reference frame.

2. You are at rest on the bus, which is moving with a constant velocity, as shown in Figure 4.10 b. Relative to the bus, your acceleration and the net force acting upon you is zero. Therefore, the first law of motion is valid on the bus. The bus moving with constant velocity can be accepted to be an inertial reference frame.

3. You and the bus are initially at rest, then the bus accelerates forwards, as shown in Figure 4.10.c. In this case, due to your inertia, you tend to remain at rest; as the bus moves forward you move backwards relative to the bus. Relative to the bus frame you are accelerating backwards. Actually you do not experience any net force and you are not accelerating (relative to the ground). It is the bus which is accelerating. Relative to the bus, you were initially at rest, then you accelerated backwards, although you did not experience any net force. So the accelerating bus cannot be accepted to be an inertial reference frame because the first law of motion is not valid in it.

As a result, any reference frame (the bus in the examples above) which is at rest or in motion with a constant velocity relative to another inertial reference frame (the Earth in the examples above) is itself an inertial reference frame. The laws of motion are valid only in inertial reference frames.

Actually the Earth cannot be an inertial reference frame. It has centripetal acceleration due to its rotational motion about its own axis and an orbital motion around the Sun. However, because these accelerations are so small, relative to the gravitational acceleration g, the Earth is can be considered to be an inertial reference frame.

In general, it is very difficult to define an inertial reference frame in the universe, since everything is in a state of motion. However, since the distant stars can be assumed to be at rest, the systems which move with a constant velocity relative to these stars can be accepted to be inertial reference frames.

## Example 4.1

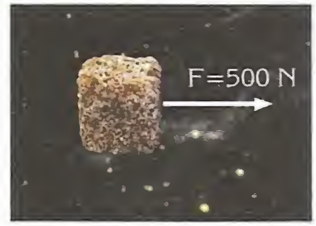


A force of 500 N acts on a load of 250 kg which is at rest in space, as shown in the figure. Find

a) the acceleration of the load,

b) the speed which the load reaches 5 s later.

c) If the force is removed 5 s after its motion starts, what will its motion be?




**Solution**

a) The force of 500 N is the net force acting on the object. From the equation of the second law of motion we can find its acceleration.

$$\vec{F} = m\vec{a}$$
$$500\ \text{N} = (250\ \text{kg})a$$
$$a = 2\ \text{m/s}^2$$

b) From the equation

$$v = v_0 + at$$
$$v = 0 + (2\ \text{m/s}^2)(5\ \text{s})$$
$$v = 10\ \text{m/s}$$

c) If the force is removed at the end of 5 s, the net force applied to the load will be zero. After that, the load will carry on moving with its final velocity of 10 m/s.

## 4.5 THE THIRD LAW OF MOTION

Newton realized that a single isolated force could not exist. Forces always occur in pairs. Thus, Newton stated the third law of motion as follows

**When one object exerts a force on a second object, the second object exerts an equal but opposite force on the first.**

The force that the first object exerts on the second is called an action force and the force that the second object exerts on the first is called a reaction force.

This law is sometimes expressed as

**To every action there is an equal and opposite reaction.**

These forces have the following properties

- Action-reaction pairs are equal in magnitude, but opposite in direction, and they act along the same line.

  For example, the force exerted by the Earth on an apple is the weight of the apple, $\vec{w}$. The reaction to this force is the force exerted by the apple on the Earth, $-\vec{w}$, where $\vec{w}$ and $-\vec{w}$ are equal in magnitude (see Figure 4.11).

  Due to the force $\vec{w}$, the apple accelerates towards the Earth. Also the Earth accelerates toward the apple due to the reaction force, $-\vec{w}$. However, since the Earth has a huge mass, its acceleration is negligibly small.

- Action-reaction pairs act on different objects. If two forces which are equal in magnitude and opposite in direction act on the same object as in Figure 4.12

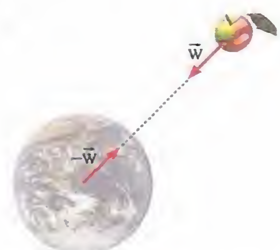



**Figure 4.11** The Earth pulls on an apple, the apple pulls on the Earth


**Figure 4.12** Two equal and opposite forces acting on the same object cancel each other.

the forces cancel each other. However, although action and reaction forces are equal in magnitude and opposite in direction, they do not cancel each other out, because they act on different objects. For example, when a force, $\vec{F}$ is exerted on a grocery cart by a shopper, as in Figure 4.13 the cart reacts on the shopper with a force of $-\vec{F}$. Since $\vec{F}$ acts on the cart and $-\vec{F}$ acts on the shopper, the cart moves.

- Action-reaction pairs are of the same type; either two contact forces or two field forces.
- Action-reaction pairs act for the same time interval.


**Figure 4.13** A shopper exerts a force on the grocery cart, and the grocery cart exerts an equal but opposite force on the shopper. Therefore these forces do not cancel each other out, since they act on different objects.

**Figure 4.14** Pushing a friend forwards means pushing yourself backwards.

**Figure 4.15** A pedestrian pushes the ground backwards; the ground then pushes the pedestrian forwards.

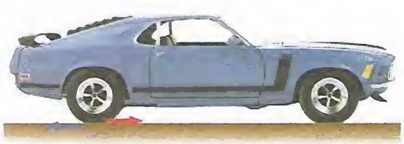


**Figure 4.16** The tyres of a car push the ground backwards; the ground then pushes the tyres forwards.

Here are some simple experiments of the third law of motion

1. Assume that you and a friend are standing on ice skates facing each other, as shown in Figure 4.14. If you push your friend away from you, you will observe that as he moves away from you, you will also move backwards because of his equal and opposite reaction force upon you.
2. You can only walk due to the reaction force of the ground on your shoes when you yourself exert an action force on the ground, as shown in Figure 4.15.
3. The tyres of a car push on the ground, thus, the ground pushes on the tyres, as shown in Figure 4.16, and the car moves.
4. Rockets also use the action and reaction principle. As a rocket pushes (action force) out a huge mass of exhaust gas downwards, as in Figure 4.17, the gas pushes (reaction force) on the rocket and the rocket moves upward.

a)

b)

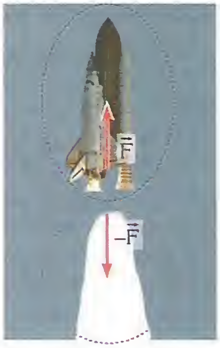



**Figure 4.17** a, b) The rocket exerts an action force on the gas and the gas exerts an equal and opposite force upon the rocket. Thus, the rocket moves forwards.

# 4.6 APPLICATIONS OF THE LAWS OF MOTION

In this section, the laws of motion will be applied to objects and systems moving with a constant acceleration under the effect of constant external forces. Sometimes the friction on surfaces may be neglected by stating that they are frictionless. The masses of ropes and pulleys in systems will also be neglected to simplify the applications of the laws of motion. Moreover, we will consider that objects do not experience circular motion nor rotate under the effect of forces. Prior to analysing the applications, the basic types of forces which will be useful in analysis will be examined.

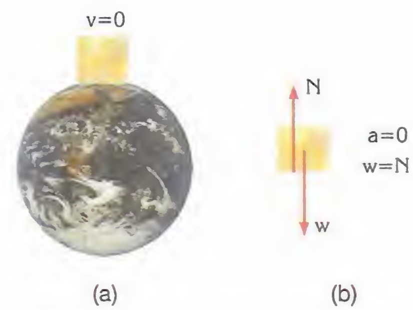



Figure 4.18 a) An object at rest on the Earth b) Its weight (w) is balanced by the force exerted by the Earth upon it, this is called the normal contact force (N).

## a. Normal Contact Force

Obviously, the acceleration of an object which is at rest on the Earth is zero, as shown in Figure 4.18.a. According to the second law of motion, $\vec{F}_{net} = m\vec{a}$, the net force acting on it is also zero. Therefore, the downward force of gravity $\vec{w}$, on the object must be balanced by an upward force. This upward force on the object is exerted by the Earth, as shown in Figure 4.18.b and is called the normal contact force, $\vec{N}$. This force is assumed to prevent the object from sinking into the Earth. Normal is a synonym for perpendicular. Normal contact force always acts perpendicular to the surface and always pushes but never pulls.

Here, we should emphasize that although the force of gravity, $\vec{w}$ and the normal contact force, $\vec{N}$ are equal in magnitude and opposite in direction, they are not action and reaction forces because they act on the same object, thus, they balance each other. Nevertheless, each of these forces must have its own pair according to the third law of motion. As shown in Figure 4.19.a, the reaction force of the normal contact force, $\vec{N}$ is the force, $-\vec{N}$ exerted by the object on the Earth. Also, the reaction force of the weight, $\vec{w}$ is the force, $-\vec{w}$ exerted by the object on the Earth, as shown in Figure 4.19.b. In order to analyse these forces clearly, the object and the Earth are separated, as shown in Figure 4.19.a and b.

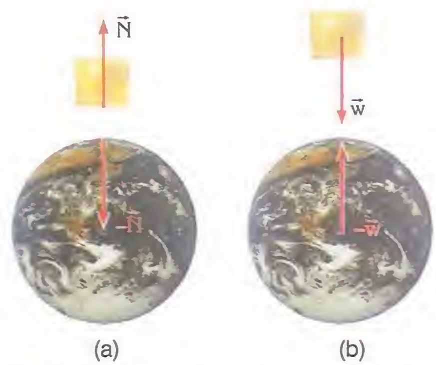



Figure 4.19 a) The reaction force of the normal contact force, $\vec{N}$ is the force, $-\vec{N}$ exerted by the object on the Earth. b) Also, the reaction force of the weight, $\vec{w}$ is the force, $-\vec{w}$ exerted by the object on the Earth.

## b. Tension Force

In our daily lives, objects can be moved by pulling them with a rope or a string. When an object is pulled by a string, the string exerts a force on the object. Such a force in a string is called a tension force and is denoted as $\vec{T}$. In other words, it is the force that each particle of the string exerts on the adjacent particles of string. Tension force acts along the string and always pulls. The ropes and the strings in the problems are assumed to be massless. Therefore, the tension in a massless rope has the same value at all points along the rope. This fact will be proven in example 4.6.

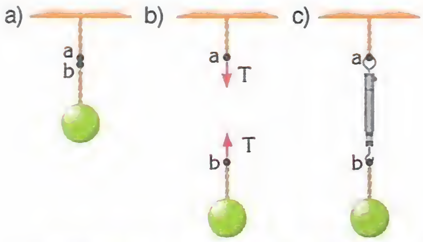



Figure 4.20 a) Two adjacent particles a and b in the string b) Each piece of the string exerts a force called the tension force T on the other. c) When the string is cut by separating adjacent particles a and b and connecting them by a dynamometer, the dynamometer reads the tension force between the particles a and b, which is the same at all points in the string.

Practically tension force can be measured as follows: Two adjacent particles a and b of a string, used to suspend an object from the ceiling are shown in Figure 4.20.a. According to the third law of motion these two pieces attract each other with the same magnitude of tension force, T (see Figure 4.20 b).

If the string is cut by separating the particles a and b and then a dynamometer placed between them, as shown in Figure 4.20.c, the value read by the dynamometer is the same as the magnitude of the tension force, T between the particles.

## c. Internal and External Forces

In order to apply the laws of motion to an object or a system, the forces need to be classified as external or internal forces. The forces acting on an object or a system from the environment are called external forces. As in Figure 4.18.b the force of gravity, $\vec{w}$ and the normal contact force, $\vec{N}$ acting on the object are two examples of external forces. Forces acting between the components of a system are called internal forces. The tension forces between the objects of a system, as shown in Figure 4.21 are internal forces, whereas $\vec{F}$ is an external force.

When the laws of motion are applied to an object or a system, only the external forces acting on them from the environment are of interest. Since internal forces (for example, tension forces as in Figure 4.21) are always equal in magnitude (T=T′) and opposite in direction, the effect of these forces on a system is zero. However, if the second law of motion is applied to each object of the system separately, as shown in Figure 4.22, tension force, which is an internal force for the whole system, becomes an external force for the object.

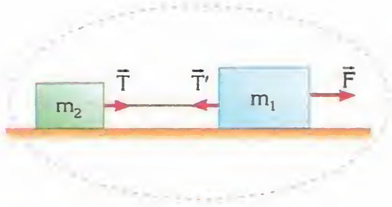



Figure 4.21 Tension forces, $\vec{T}$ and $\vec{T}'$, are internal forces, whereas the applied force $\vec{F}$ is an external force.

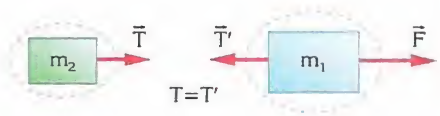
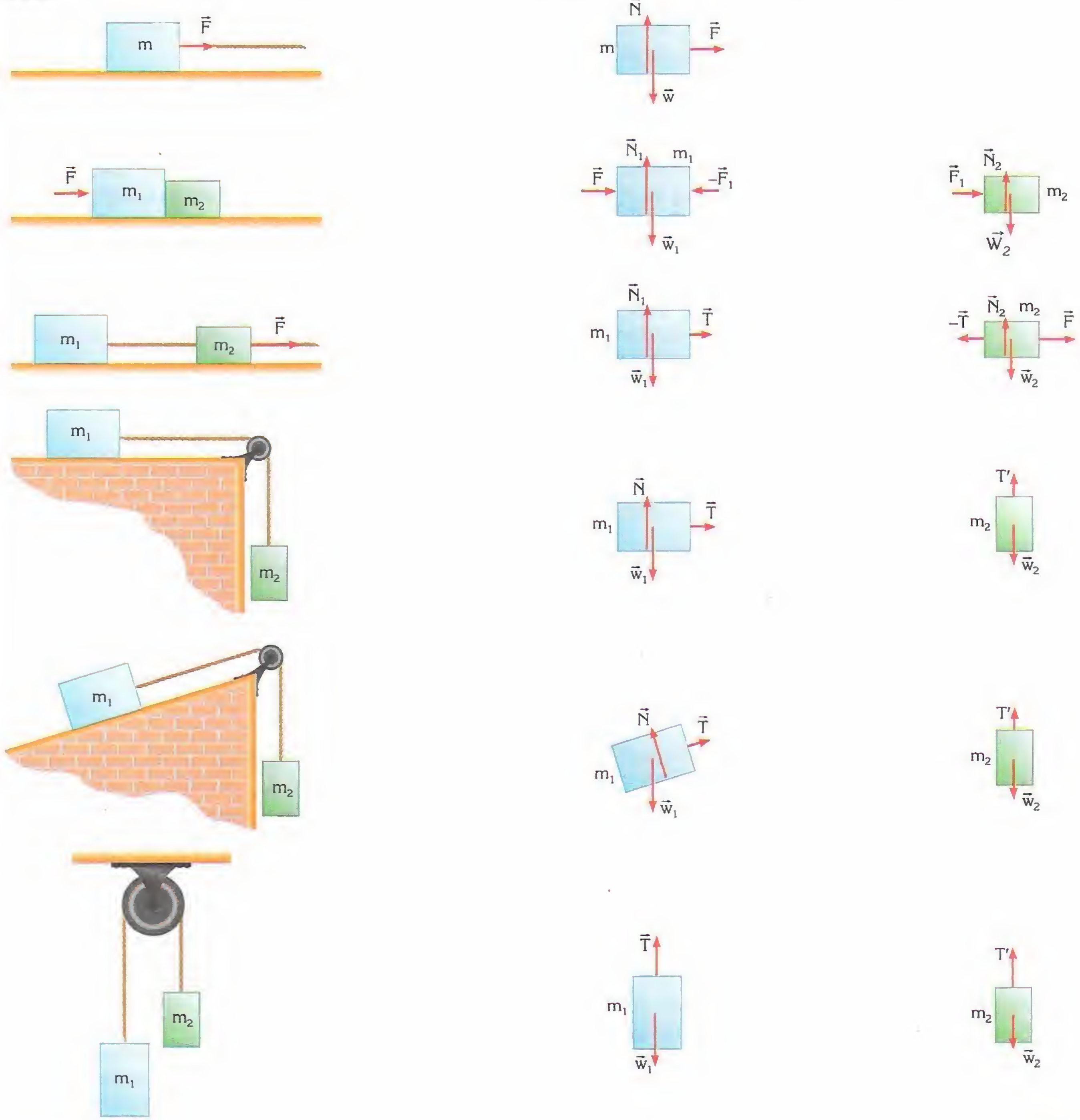



Figure 4.22 Since the second law of motion is applied to each object of the system separately, $\vec{T}$ and $\vec{T}'$ are regarded as external forces for the masses $m_2$ and $m_1$, respectively, whereas the applied force $\vec{F}$ is always taken to be an external force.

## Free-Body Diagrams

In dynamics problems it is very important to analyse the forces acting on systems or objects correctly. Therefore, the system or the object whose state of motion is to be analysed must be isolated from the outside environment and all the forces acting on it must be shown clearly. This method of illustration is called a "free-body diagram". Some free-body diagrams of applied forces are shown below.



## Problem Solving Strategy

Here is a helpful procedure to aid in the application of the laws of motion to problem solving

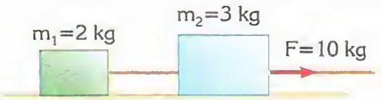
1. Draw a simple diagram of the system
2. Draw the free-body diagram of the system or the object whose motion is to be analysed, showing all external forces acting on it.

   Do not forget that some of the internal forces for a system may be external forces for an object in that system. Also, do not show the forces exerted by an object or a system on its environment in drawing the free body diagram.
3. Select a suitable inertial reference frame. Position the origin and the orientation of the coordinate axes and find the components of the forces along these axes. Then apply the second law of motion,

   $\vec{F}_{net}=m\vec{a}$, to each component of the system.
4. If there are more than one unknown, there must be the same number of equations as the number of unknowns. The unknowns, can be obtained by solving the equations.

## Example 4.2 — Blocks pulling each other on a frictionless horizontal plane

Masses $m_1$ and $m_2$ attached to each other with a string, are moving on a smooth horizontal surface under the effect of a horizontal force of 10 N, as shown in the figure. If $m_1$ = 2 kg and $m_2$ = 3 kg, find

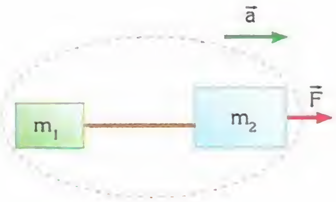



a) the acceleration of the system,

b) the value of the tension in the string.

### Solution

a)




Taking the system of masses attached to each other as a whole, apply the second law of motion to the system.

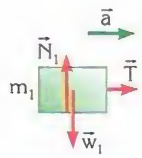
The net force acting on the system is force $\vec{F}$. Since the weights of the objects and the normal contact forces on the objects are perpendicular to the direction of motion, they do not affect the motion. Also,since the tension in the rope is an internal force,it is not taken into account. Therefore, the acceleration of the system is;

$$\vec{F}_{net}=m\vec{a}$$
$$F = (m_1 + m_2)a$$
$$10\text{ N} = (3\text{ kg} + 2\text{ kg})\ a$$
$$a = 2\text{ m/s}^2$$

b) In order to obtain the tension in the string, apply the second law of motion to one of the masses of the system.

### Solution 1

Vertical forces, $\vec{N}_1$ and $\vec{w}_1$, on $m_1$ cancel each other out, also since there is no friction, they do not affect the motion.




The net force acting on mass $m_1$ is the tension $\vec{T}$ in the string. Hence, applying the second law of motion to mass $m_1$,

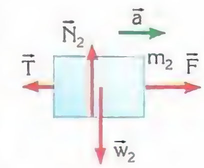
$$\vec{F}_{net}=m\vec{a}$$

$$T = m_1 a$$

$$T = (2\ \text{kg})(2\ \text{m/s}^2) \quad \text{thus} \quad T = 4\ \text{N}$$

### Solution 2

Find the tension in the string, this time applying the second law of motion to mass $m_2$. Since the system accelerates in the direction of $\vec{F}$, the net force acting on mass $m_2$ is F–T. The vertical forces do not affect mass $m_2$. Applying the second law of motion




$$\vec{F}_{net}=m\vec{a}$$

$$F - T = m_2 a$$

$$10\ \text{N} - T = (3\ \text{kg})(2\ \text{m/s}^2)$$

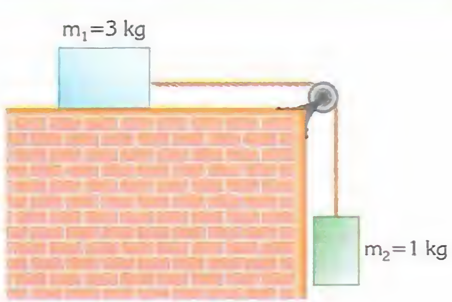
$$T = 4\ \text{N}$$

## Example 4.3 Two blocks connected by a frictionless pulley

A frictionless system consisting of masses $m_1$=3 kg and $m_2$=1 kg is shown in the figure. If the system is released from rest, find

a) the acceleration the system gains

b) the tension in the string. (g = 10 N/kg)




### Solution

a) The weight, $\vec{w}_1$ of the mass $m_1$ is balanced by the normal contact force $\vec{N}_1$. So, the system moves only under the effect of the weight of mass $m_2$. Applying the second law of motion to the system, its acceleration is obtained.

$$\vec{F}_{net}=m\vec{a}$$

$$m_2 g = (m_1 + m_2)a$$

$$(1\ \text{kg})(10\ \text{N/kg}) = (3\ \text{kg} + 1\ \text{kg})a$$

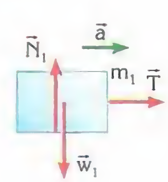
$$a = 2.5\ \text{m/s}^2$$

b) The tension in the string is obtained by applying the second law of motion to one of the masses.

### Solution 1

For the mass $m_1$

The tension force has an affect on the motion of $m_1$.




$$\vec{F}_{net}=m_1\vec{a}$$

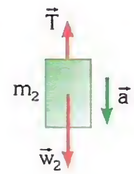
$$T = m_1 a$$

$$T = (3\ \text{kg})(2.5\ \text{m/s}^2)$$

$$T = 7.5\ \text{N}$$

## Solution 2

This time apply the second law of motion to $m_2$ to find T




$$\vec{F}_{net} = m_2\,\vec{a}$$

$$w_2 - T = m_2\,a$$

$$m_2\,g - T = m_2\,a$$

$$(1\ kg)(10\ N/kg) - T = (1\ kg)(2.5\ m/s^2)$$

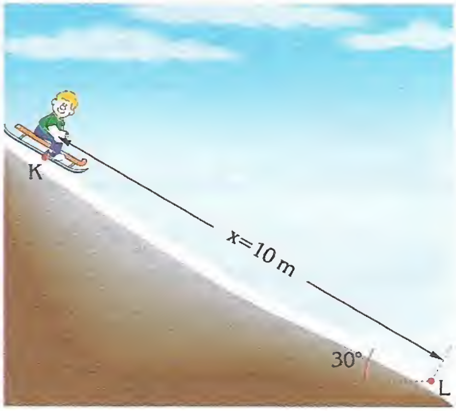
$$T = 7.5\ N$$

## Example 4.4 — A child on a sledge

A child starts riding his sledge from point K, on a hill covered with snow and inclined at an angle of 30°, as shown in the figure. Calculate

a) the acceleration he will gain

b) the time taken for him to reach point L. ($g = 10$ N/kg)




### Solution

a) It is convenient to choose the x-y coordinate system with the x axis along the inclined plane and the y axis perpendicular to it.

We assume that the surface is frictionless. Applying the second law of motion to the sledge on each axis, gives

$$\vec{F}_{net} = m\vec{a}$$

$$\Sigma\vec{F}_x = m\vec{a}_x$$

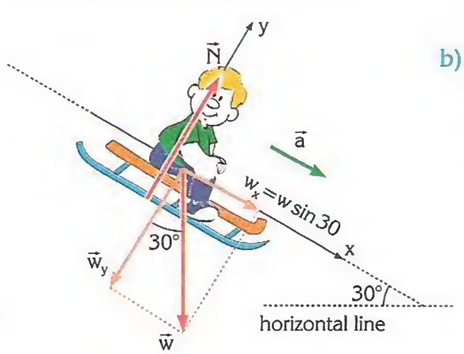
$$w_x = ma_x$$

$$mg\sin 30° = ma_x$$

$$g\sin 30° = a_x$$

$$(10\ m/s^2)0.5 = a_x$$

$$a_x = 5\ m/s^2$$




$$\Sigma\vec{F}_y = m\vec{a}_y$$

$N - w_y = ma_y$ where $a_y = 0$

$N = w_y = mg\sin 30$ that is, $\vec{w}_y$ is balanced by $\vec{N}$.

Since $a_y = 0$, $a_x = a = 5\ m/s^2$

b) Solve this part of the problem using kinematic equations. Since the initial velocity is $v_0 = 0$ using the equation;

$$\Delta x = v_0 t + \frac{1}{2}at^2$$

$$10\ m = 0 + \frac{1}{2}(5\ m/s^2)t^2$$

$$t = 2\ s$$

## 4.7 FORCE OF FRICTION

So far the effects of friction have been ignored, but it must be taken into account in most practical situations. When a body slides over the surface of another, its motion is always opposed by a retarding force that resists this motion. This force is called the force of friction. Forces of friction are very important to us in our everyday lives. Without friction we could not walk, move, stop or turn corners. We would not be able to hold objects, for example a pencil, even if we could hold it, it would not write, since writing also relies on friction, we would not be able to drive cars.

Suppose there is a block at rest on a horizontal surface, as shown in Figure 4.23.a A spring balance (a dynamometer) is attached to it to measure the force needed to set the block in motion. Before applying any force on the block, no force of friction acts upon it. When the block is pulled with a small horizontal force $\vec{F}$, it still remains at rest. From the second law of motion, it can be concluded that there must be another force acting on the block which opposes and balances the applied force $\vec{F}$. This force is the friction force, $\vec{f}$ exerted on the block by the surface. If the applied force is increased, the friction force acting on the block also increases. As long as the block remains at rest, the magnitudes of the applied force and the force of friction are equal, $F=f_s$, as shown in Figure 4.23.b,c,d. When the block is not moving, the friction force that it experiences is called the force of static friction, $f_s$. When the applied force reaches a certain value at which the block is just about to move, $f_s$ also reaches its maximum value, $f_{s,max}$, as shown in Figure 4.23.d. Once the motion has started, as shown in Figure 4.23.e, the applied force starting the motion accelerates the block. By decreasing this force it is possible to keep the block in motion with a constant velocity ($a=0$), as shown in Figure 4.23.f. Thus, the force required to start the motion of the block is slightly greater than the force required to keep it in uniform motion.

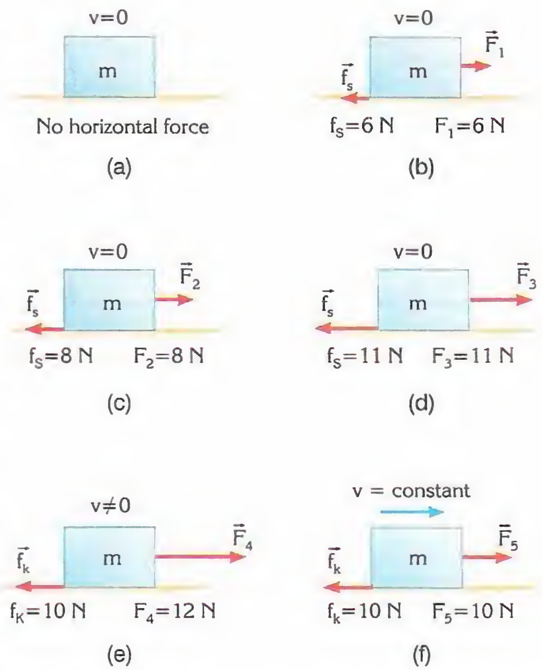



**Figure 4.23** Static and kinetic friction forces which act on the block of mass m

When the block is in motion, the friction force acting on it is called the force of kinetic friction, $f_k$.

If we apply a force F on the block greater than $f_k$, the block accelerates. If this force is removed, the force of kinetic friction acting on the block brings it to rest.

We can summarise the experimental observations with Figure 4.24, as follows, note that $\vec{F}$ is the applied force on the block

- $F = f_s$ as long as the block is at rest
- At the instant $F= f_{s,\ max}$ it starts sliding
- If $F = f_k$ it moves with a constant velocity
- If $F>f_k$ it accelerates
- We should emphasize that $f_{s,max}>f_k$

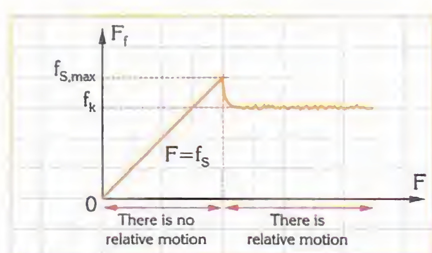



**Figure 4.24** A graph showing the values of static and kinetic friction forces acting on an object which starts to move from rest.

| Materials | Static friction, $\mu_s$ | Kinetic friction, $\mu_k$ |
|---|---|---|
| Steel on steel | 0.74 | 0.57 |
| Aluminum on steel | 0.61 | 0.47 |
| Zinc on cast iron | 0.51 | 0.44 |
| Copper on cast iron | 1.05 | 0.29 |
| Glass on glass | 0.94 | 0.40 |
| Copper on glass | 0.68 | 0.53 |
| Teflon on Teflon | 0.04 | 0.04 |
| Ice on ice | 0.1 | 0.03 |
| Rubber on concrete | 1.0 | 0.8 |
| Synovial joints in human body | 0.01 | 0.003 |

**Table 4.1** Some reported values of the coefficients of friction. Notice that the smallest one is that in the synovial joints of the human body, which are vital for human life.

Experiments have revealed the following empirical rules of friction

1. Friction forces are always parallel to the surface of contact and opposite to the direction of motion or intended motion.
2. The force of static friction can have values between 0 and $\mu_s N$

$$f_s \leq \mu_s N$$

where $f_s$ is the magnitude of the force of static friction, $\mu_s$ is the coefficient of static friction and N is the magnitude of the normal force. The equality holds only when $f_s$ has its maximum value.

3. The force of kinetic friction is given by

$$f_k = \mu_k N$$

where $f_k$ is the magnitude of the force of kinetic friction, $\mu_k$ is the coefficient of kinetic friction and N is the magnitude of the normal force.

The values of $\mu_s$ and $\mu_k$ depend on the nature of both surfaces in contact. Some average values of $\mu_s$ and $\mu_k$ for different pairs of materials are shown in Table 4.1. Generally $\mu_k$ is less than $\mu_s$. That is why it is easier to keep the block moving than to start it moving.

4. The coefficient of kinetic friction is quite independent of the relative speed of the surfaces in contact.
5. The coefficients of friction are nearly independent of the contact area between the surfaces.

## What Causes Friction?

No matter how smooth the surfaces of objects appear to be, in reality they are rough and they have irregularities on their surfaces, as shown in Figure 4.25. However, the friction does not arise from only mutual contact of irregularities on the surfaces. Experiments show that the real causes of friction are "the electrostatic forces" which occur between the molecules of the surfaces. When the flat surfaces of two objects are placed in contact, the actual (microscopic) area of contact is much smaller than the apparent (macroscopic) area of contact. It is rather like turning Switzerland upside down and placing it on top of Austria. Only the tips of mountains will touch. The actual area of contact is proportional to the normal force. When the normal force is constant, actual area of contact is also constant. The actual contact area remains the same even when the apparent contact area is reduced because increased normal force per unit actual area brings more molecules closer to interact. That is to say, the number of molecules forming bonds between the surfaces in both cases of having small and large apparent contact area is the same. Consequently friction force is the same. As a block slides over a surface, the bonds between the surfaces form and break.

Up to a certain extent, smoothing a surface removes the irregularities and decreases friction. However, it brings more molecules, capable of bonding, closer and in this way it actually increases friction.

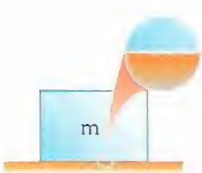
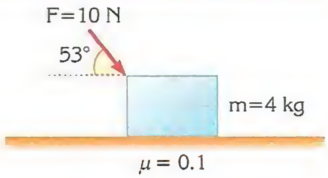



**Figure 4.25** Friction force does not arise only from the mechanical interactions of irregular surfaces. It actually occurs due to the bonds formed between the molecules at high points on the surfaces

## Example 4.5 A block on a rough surface



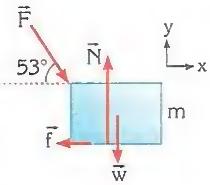

A 4 kg block is stationary on a surface. The coefficient of kinetic friction between the block and the surface is 0.1. If a force of 10 N is applied to the block, as shown in the figure, find

a) the friction force acting on the block,

b) the acceleration gained by the block. (Take $\sin 53° = 0.8$; $\cos 53° = 0.6$; $g = 10$ N/kg )

### Solution

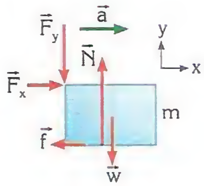
a) Let us first draw the free-body diagram of forces acting on the block.




The block does not accelerate along the y axis. Applying the second law of motion to the y axis, the normal contact force can be found, and then the frictional force.

$$\vec{F}_{net} = m\vec{a}$$

$$\Sigma\vec{F}_y = m\vec{a}_y$$




$$N - (w + F_y) = ma_y \quad \text{where } a_y = 0$$

$$N = (w + F_y)$$

$$f_k = \mu N = \mu(w + F_y)$$

$$f_k = \mu(mg + F \sin 53°)$$

$$f_k = 0.1[(4 \text{ kg})(10 \text{ N/kg}) + (10 \text{ N})\, 0.8]$$

$$f_k = 4.8 \text{ N}$$

b) The block moves in the direction of the net force, which acts along the horizontal direction. Since $F_x > f_k$, the block accelerates in the direction of $\vec{F}_x$. Applying the second law of motion along the x axis, the acceleration of the block is found.

$$\vec{F}_{net} = m\vec{a}$$

$$\Sigma\vec{F}_x = m\vec{a}_x$$

$$\text{where} \quad \vec{a}_x = \vec{a}$$

$$F_x - f_k = ma$$

$$F \cos 53° - f_k = ma$$

$$(10 \text{ N})0.6 - 4.8 \text{ N} = 4a$$

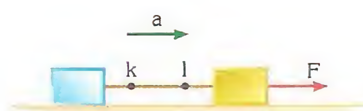
$$a = 0.3 \text{ m/s}^2$$

## Example 4.6 Tension force


Assume that the two masses connected to each other with a massless string are accelerating in the direction of force F, as shown in the figure. Prove that the tension in a string has the same value at all points along the string.

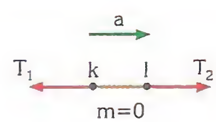
### Solution

Let's take any two pieces, k and l in the string and cut the string between these points and then apply the second law of motion to the segment between these two points. If the forces acting on the pieces k and l, due to the neighbouring pieces, are labeled $T_1$ and $T_2$, from the second law of motion




$$\vec{F}_{net} = m\vec{a}$$

$$\vec{T}_2 - \vec{T}_1 = m\vec{a} \quad \text{where } m=0\text{, because the string is massless.}$$




$$\vec{T}_2 = \vec{T}_1$$

This result shows that the tension force is the same at all points in the string.



## Terminal Velocity

Figure 4.26 At the instant the air drag balances a skydiver's weight, she descends with a constant velocity called the terminal velocity.

Figure 4.27 When she opens her parachute, the surface area increases therefore air drag increases. Eventually the air drag balances her weight and she attains a terminal velocity.

So far we have studied the motions of objects in the absence of air resistance. Air resistive forces acting on freely falling objects have not been considered.

Actually any object in motion experiences an air resistive force (commonly called air drag) arising from collisions between the air particles and the moving object.

The air drag depends on velocity, v, the shape of the object, the surface area, the material of which the surface is made and the density of the air. Depending on the velocity of the object, a constant of proportion k is obtained from the combination of all the other factors. Thus, the following equation is derived for the air drag.

$$F_{res} = kv^2$$

Consider the following example; when a skydiver steps out of an aircraft, before opening her parachute, her velocity and thus the air drag acting on her increases. Eventually the air drag balances her weight, as shown in Figure 4.26. At that instant, since the net force acting upon her is zero, she starts descending with a constant velocity called the **terminal velocity**.

When the skydiver opens her parachute, the surface area and thus the constant of proportionality, k, increases. This results in an increase in the air drag to a value greater than her weight thus, her velocity decreases. This causes the air drag acting on her to decrease according to the equation above. Eventually, once again the air drag balances her weight and she attains a terminal velocity for the second time. The skydiver continues to descend at this constant velocity, as shown in Figure 4.27.

### Example 4.7 — Terminal velocity

If the terminal velocity of a $3.4\times10^{-5}$ g rain drop in the air is 9 m/s, find the constant of proportionality, k, for air.

**Solution**

The rain drop descends with a terminal velocity because the air drag balances its weight.

$$F_{res} = w$$

$$kv^2 = mg$$

$$k = \frac{mg}{v^2} = \frac{(3.4\times10^{-8}\ \text{kg})(10\ \text{N/kg})}{81\ \text{m}^2/\text{s}^2} = 0.42\times10^{-8}\ \text{Ns}^2/\text{m}^2$$

## Weightlessness

A person standing on the ground is acted on by both the downward force of gravity and the upward reaction force (normal contact force) of the ground. The gravitational force acting upon the person is called **actual weight**. The normal contact force is called **apparent weight**. That is, the force an object exerts on a surface is its apparent weight. Similarly an object suspended with a string is acted upon by two forces: The downward force of gravity and the tension from the string. The force of gravity is the actual weight, whereas, the tension force is the apparent weight of the object.

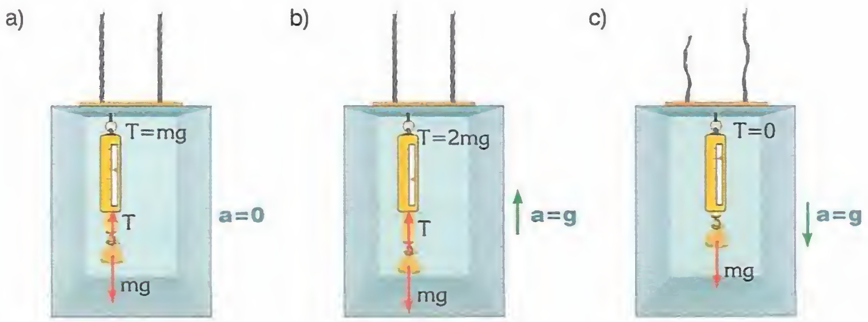
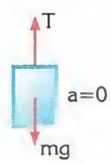



**Figure 4.28** a) An object at rest applies a force on the dynamometer equal to its weight. b) In a lift accelerating upwards at a=g, the object's apparent weight is doubled. c) In a freely falling lift the object experiences weightlessness.

Let's analyse the apparent weight of a bag in a falling elevator. As in Figure 4.28.a, when the elevator is at rest (a=0), the dynamometer reads the actual weight of the bag which is mg. This reading (T) can be calculated by applying the second law of motion.



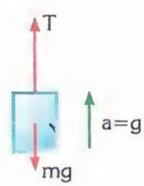

$$\vec{F}_{net} = m\vec{a}$$

$$T - mg = 0$$

$$T = mg \qquad \text{(real weight)}$$

The spring force, T in the dynamometer balances the weight of the bag and the dynamometer reads a value indicating the actual weight of the bag.

As the elevator rises with an acceleration of g, as in Figure 4.28.b,



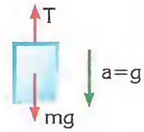

$$\vec{F}_{net} = m\vec{a}$$

$$T - mg = ma$$

$$T = mg+mg$$

$$T = 2mg$$

The dynamometer reads 2mg which is twice the actual weight. That is, the apparent weight is twice the actual weight.




If the elevator is in free fall (for example, if the cables break), the elevator and everything in it fall at the gravitational acceleration, a=g, as shown in Figure 4.28.c

$$\vec{F}_{net} = m\vec{a}$$

$$mg - T = ma$$

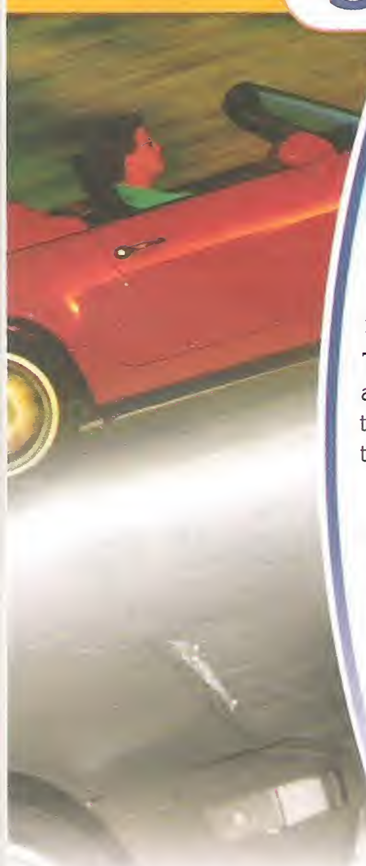
$$mg - T = mg$$

$$T = 0$$

That is, the dynamometer reads 0. The bag seems weightless.

This phenomenon is called **weightlessness**.

# Summary



**The first law of motion** states that

If the net force acting on an object is zero

- If it is at rest, it will remain at rest.
- If it is moving, it keeps on moving at a constant velocity (a constant speed in a straight line)

The tendency of an object to resist any change in its state of motion is called **inertia**.

**The second law of motion** states that the acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass.

$$\vec{F}_{net} = m\vec{a}$$

The force exerted by the Earth on an object is called the **weight** of that object. It is directed towards the centre of the Earth and its magnitude changes with location. It is equal to the product of the mass of the object and the gravitational acceleration g at that location.

$$\vec{w} = m\vec{g}$$

**The third law of motion** states that for every action there is an equal and opposite reaction. Thus, an isolated force can never exist in nature.

The **force of static friction** can have values between 0 and $\mu_s N$

$$f_s \le \mu_s N$$

where $f_s$ is the magnitude of the force of static friction, $\mu_s$ is the

coefficient of static friction and N is the magnitude of the normal contact force. The

equality holds only when $f_s$ has its maximum value.

The **force of kinetic friction** is given by

$$f_k = \mu_k N$$

where $f_k$ is the magnitude of the force of kinetic friction, $\mu_k$ is the

coefficient of kinetic friction and N is the magnitude of the normal contact force.

The values of $\mu_s$ and $\mu_k$ depend on the nature of both surfaces in contact.

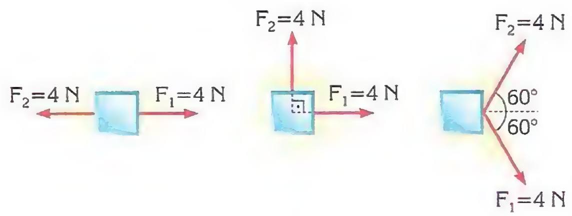
# QUESTIONS AND PROBLEMS

### 4.1 The Concept of Force
### 4.2 Net Force

1. What is a force?

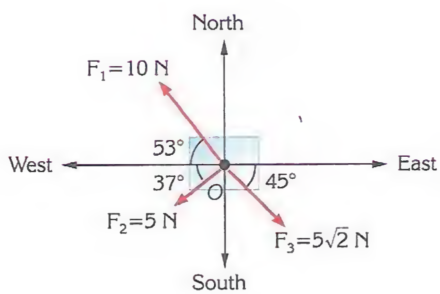
2. What is a "net force"?

3. The magnitudes and the directions of some forces acting on an object on a frictionless surface are shown in the figure. Find the net force in each case;




(a) (b) (c)

4.




In the diagram above, three forces act on an object. Find the net force acting on the object.
($\sin 53° = \cos 37° = 0.8$, $\sin 37° = \cos 53° = 0.6$,

$\sin 45° = \cos 45° = \frac{\sqrt{2}}{2}$ )

### 4.3 The First Law of Motion
### 4.4 The Second Law of Motion
### 4.5 The Third Law of Motion

5. Explain the first law of motion and inertia

6. If a body is stationary, can we say that there is no force acting on it?

7. If no net force acts on a body, is it possible for it to move?

8. Explain the second law of motion

9. Express the unit Newton in terms of base SI units.

10. If there is only one force acting on a body, can it be at rest?

11. If the acceleration of a body is zero, can we say that no force acts on it?

12. Is the motion of bodies always in the same direction as that of the net force?

13. Can the direction of the net force acting on an object be opposite to that of its acceleration?

14. To which law of motion is the operation of car seat belts related?

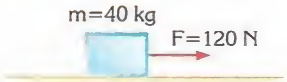
15. Which forces act on an apple when it
   a) remains on a tree?
   b) falls from a tree?
   c) remains on the ground?

16. According to the third law of motion, when you push somebody, he does not have a right to complain about you. Why?

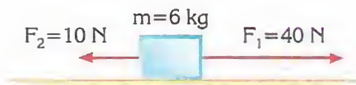
17. Prove that $N/kg = m/s^2$

**18.** Find the acceleration of an object experiencing a force $F=120$ N, as shown in the figure.

(The surface is frictionless).



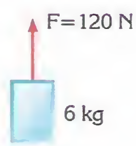

**19.**




Find the acceleration of an object on a smooth surface, as shown in the figure.

**20.** If the mass of the object in problem 3 is 2 kg, find the acceleration of the object for each case.

**21.** Find the acceleration of the object shown in the figure. (Take $g=10$ N/kg)

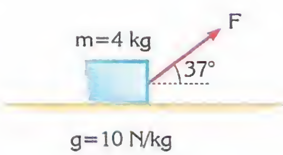



**22.** Find the acceleration of the mass on a smooth surface shown in the figure when

a) $F=50$ N

b) $F=100$ N

($\sin 37°=0.6$; $\cos 37°=0.8$)




**23.** Two perpendicular forces act on a 400 g object. The object moves with an acceleration, $a=25$ m/s$^2$. If $F_1=8$ N find the other force? (Assume no gravity acts)

**24.** What force is needed to stop a truck of mass 4 tons moving at a velocity of 36 km/h in 50 m?

**25.** An object of mass 1g is moving under a force of 1N. Find its acceleration.

**26.** A 2-kg object accelerates from rest to 5 m/s in 0.4 s. What is the net force on the object?

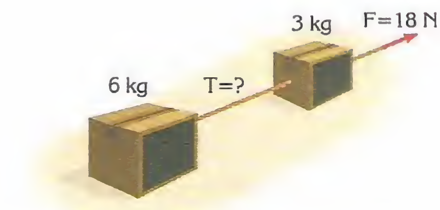
**27.** A force of 20 N acts on a 5-kg object, initially at rest. What distance does the object travel in 3 s? (assume no gravity acts)

**28.** A 10 g-bullet leaves an 80 cm long barrel of a rifle at a velocity of 400 m/s. What is the average force on the bullet while it is in the rifle?

**29.** A man weighs 800 N on Earth. Find his weight and mass on Jupiter where $g=26$ N/kg.

(on Earth $g=10$ N/kg)

**30.**

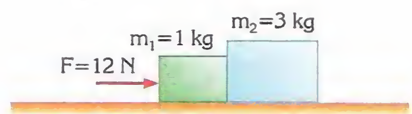



Two boxes having masses of 3 kg and 6 kg, attached to each other by a rope, rest on a horizontal smooth surface. If the 3-kg box is pulled by a force of 18 N, as shown in the figure, find

a) the acceleration of the boxes

b) the tension in the rope between the boxes.

**31.**




The system in the figure moves under the effect of a force of 12 N on a smooth surface. If $m_1=1$ kg and $m_2=3$ kg, find

a) the acceleration of the system,

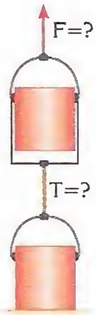
b) the force moving the mass $m_2$.

**32.** On a building site, two buckets of paint of masses 10 kg each are attached to each other by a rope in the vertical as shown in the figure. Find the force F and the tension in the rope between the buckets when

a) the buckets are at rest

b) the buckets are pulled upwards with an acceleration of 1 $m/s^2$

(Neglect the weight of the rope.)

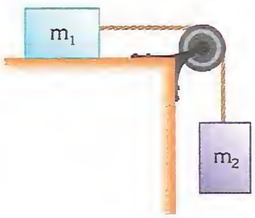



**33.** A frictionless system consisting of masses $m_1=3$ kg and $m_2=1$ kg is shown in the figure. If the system is released from rest, find

a) the acceleration of the system,

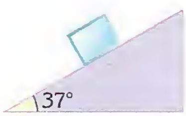
b) the tension in the string.

(Take g=10 N/kg)

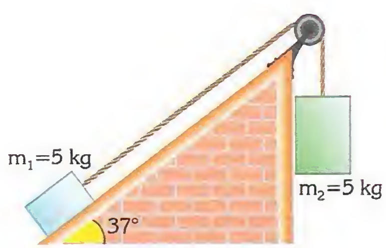



**34.** If the inclined plane, shown in the figure, is frictionless, find the acceleration of the object.

($g=10$ $m/s^2$; $\sin 37°=0.6$, $\cos 37°=0.8$)




**35.**

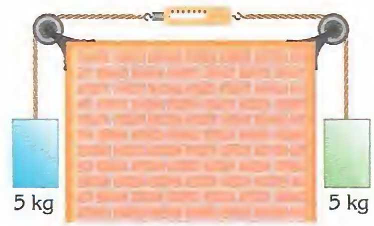



Two blocks, 5 kg each, are attached to each other with a string. They are placed on an inclined frictionless plane, as shown in the figure, then released. Find

a) the acceleration gained by the system

b) the tension in the string. ($\sin 37°=0.6$; $\cos 37°=0.8$)

**36.**

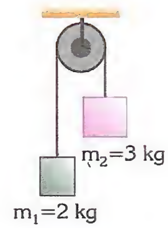



Two blocks, of mass 5 kg each, are attached to two ends of a dynamometer by the strings shown in the figure. What is the force displayed on the dynamometer? (Take g=10 N/kg)

**37.** Neglecting the friction in the system shown in the figure find

a) the acceleration of the system

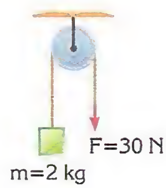
b) the tension in the string




**38.** Neglecting the friction in the system shown in the figure find

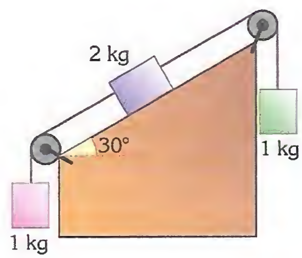
a) the acceleration of the object

b) the tension in the string

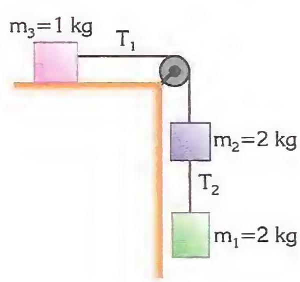



**39.** If the system shown in the figure is frictionless, what is the acceleration of the system?

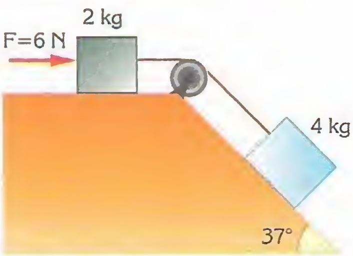
($\sin 30°= 0.5$)




**40.** What is the ratio of the tensions $\frac{T_1}{T_2}$ in the strings connecting the masses moving in the smooth system shown in the figure?



**41.**



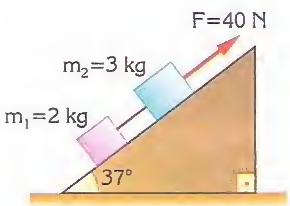

For the frictionless system shown in the figure, find

a) the acceleration of the system

b) the tension T in the string.

(sin37=0.6; cos37=0.8)

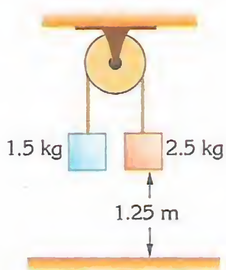
**42.** If masses $m_1$ and $m_2$ on the inclined frictionless surface are pulled by a force F=40 N, as shown in the figure,




a) what is the acceleration of the system

b) find the tension in the string connecting the masses. (sin37=0.6, cos37=0.8)

**43.** The masses in an Atwood machine are 1.5 kg and 2.5 kg (See the figure). If the masses are at a height of 1.25 m when the system is released

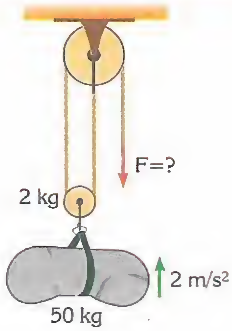



a) how many seconds does it take for the greater mass to reach the ground?

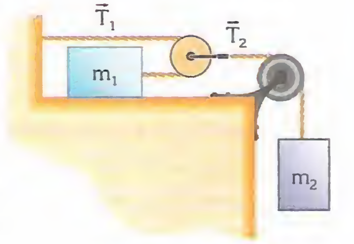
b) What is the velocity of the smaller mass when the greater one strikes the ground?

c) If this experiment were carried out on the Moon, how many seconds would it take for the 2.5 kg mass to fall? ($g_E$=10 N/kg·$g_M$=$g_E$/6)

**44.** The cement bag shown in the figure has a mass of 50 kg and the small pulley has a mass of 2 kg. If the cement bag is pulled upwards with an acceleration of 2 m/s$^2$, find the force F pulling the rope.




**45.** The masses $m_1$=1 kg and $m_2$=6 kg, in the system shown in the figure, are attached to each other with weightless pulleys and strings. If the pulleys and the surface are smooth

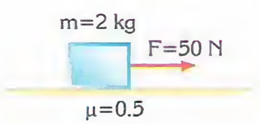



a) what are the accelerations of $m_1$ and $m_2$?

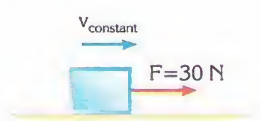
b) what are the tensions in the strings?

## 4.7 Force of Friction

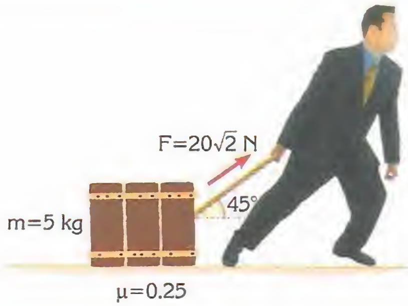
**46.** Find the acceleration of the object of mass m=2 kg if a horizontal force of 50 N acts on it, on a rough surface where the coefficient of kinetic friction is 0.5, as shown in the figure. (Take g =10 N/kg)




**47.** When a horizontal force of 30 N acts upon a 5kg-object on a rough surface, as shown in the figure, the object moves with a constant velocity. Find the coefficient of kinetic friction, $\mu_k$ between the object and the surface. (Take g=10 N/kg)

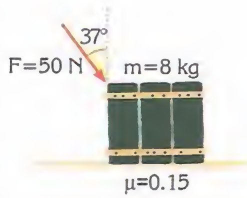


**48.**




A child pulls a 5 kg-object on a rough horizontal surface with an angle of 45°, as shown in the figure. If the coefficient of kinetic friction between the object and the surface is 0.25, what is the acceleration of the object? ($g=10$ N/kg, $\sin 45°=\cos 45°=\sqrt{2}/2$)

**49.** A force of 50 N is applied to a mass of 8 kg at an angle of 37° to the vertical, as shown in the figure. If the coefficient of kinetic friction between the surface and the object is 0.15, find the acceleration of the object? ($\sin 37°=0.6$, $\cos 37°=0.8$)



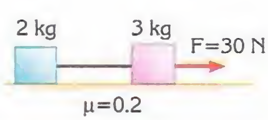

**50.** A train having a velocity of 72 km/h, stops in 4 seconds when all the wheels are blocked. What is the coefficient of kinetic friction? (Take $g=10$ N/kg)

**51.** An ice skater moving at 10 m/s slides to a halt in 100 m on an icy surface. What is the coefficient of kinetic friction between the ice and the skates?

**52.** What is the minimum stopping distance for a car moving at 72 km/h if the coefficient of kinetic friction between the tyres and the road is 0.8?

**53.** What is the effect of a car's mass on its minimum stopping distance?

**54.** Two objects of mass 2 kg and 3 kg are moved by a force $F=30$ N along a horizontal surface, as shown in the figure. If the coefficient of kinetic friction between the objects and the surface is $\mu=0.2$

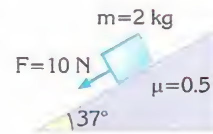



a) what is the acceleration of the system?

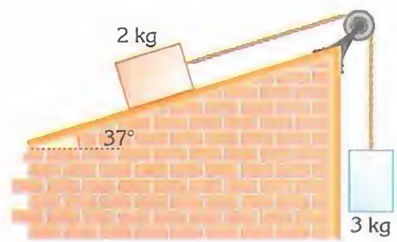
b) find the tension in the string connecting the objects.

**55.** In the system shown in the figure the masses of the objects are $m_1=0.8$ kg, $m_2=0.5$ kg, $m_3=1.2$ kg. If the coefficient of kinetic friction is 0.1 between the surface and mass $m_2$, what is the acceleration of the system? (Take $g=10$ N/kg)


**56.** Find the acceleration of the object on the rough inclined plane shown in the figure ($\sin 37°=0.6$; $\cos 37°=0.8$)

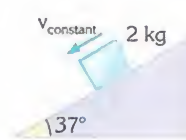



**57.**

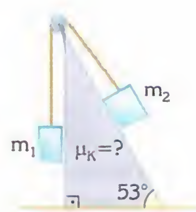



A 2 kg-container on an inclined plane at an angle of 37° with the horizontal is attached to another object of mass 3 kg, as shown in the figure.
If the coefficient of kinetic friction between the surface and the 2 kg object is 0.5, find the acceleration of the system and the tension in the string
($\sin 37°=0.6$; $\cos 37°=0.8$)

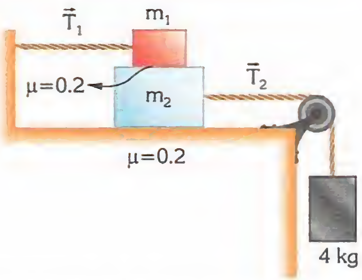
**58.** If the mass shown in the figure is moving down the inclined plane with a constant velocity, find the coefficient of kinetic friction between the inclined plane and the mass. ($\sin 37°=0.6$; $\cos 37°=0.8$)




**59.** Masses $m_1$ and $m_2$ are moving at a constant velocity on an inclined plane, as shown in the figure. If $m_1=m_2=2$ kg, what is the coefficient of kinetic friction between mass $m_2$ and the surface?
($\sin 37°=\cos 53°=0.6$; $\cos 37°=\sin 53°=0.8$)



**60.**

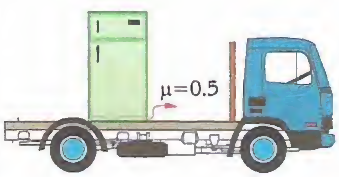



In the system shown in the figure, $m_1=3$ kg, $m_2=5$ kg. The coefficient of kinetic friction between $m_1$–$m_2$ and $m_2$–surface is 0.2.

a) find the acceleration of mass $m_2$.

b) find the tensions in the strings.

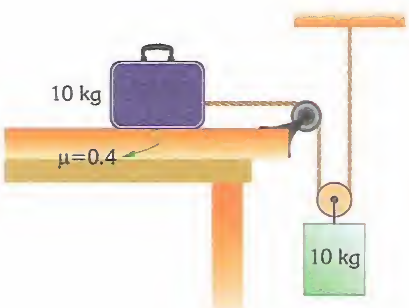
**61.** The coefficient of kinetic friction between the base of a refrigerator standing on the back of a lorry and the body of the lorry is 0.5.




a) What is the maximum acceleration that the lorry can gain in order that the refrigerator will not slide?

b) 6 s after the lorry starts to accelerate at 2.5 m/s$^2$, at what distance should it stop after braking so that the refrigerator will not slide?

**62.**

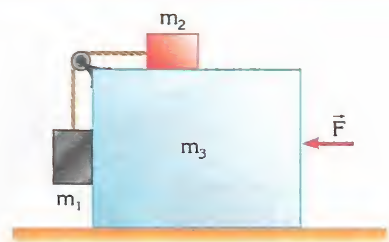



The coefficient of kinetic friction between the tool bag and the table shown in the figure is 0.4. If the pulleys are weightless, find

a) the acceleration gained by the tool bag

b) the tension in the rope after the system is released.

**63.**

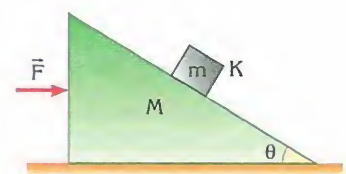



The surfaces in the system shown in the figure are frictionless and $m_1=2$ kg, $m_2=1$ kg and $m_3=5$ kg. For masses $m_1$ and $m_2$ to be able to stay stationary on $m_3$

a) what should the acceleration of $m_3$ be?

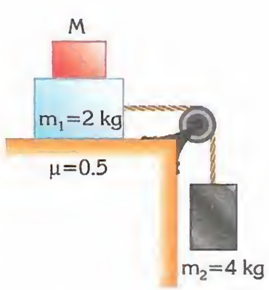
b) what magnitude should force F have?

**64.**




A smooth frictionless system is shown in the figure. For object K to remain stationary on the inclined plane, pushed by a force F, what should the magnitude of F be if M=4 kg, m=1 kg, g=10 N/kg and $\theta=37°$. ($\sin 37°=0.6$, $\cos 37°=0.8$)

**65.** In the system shown in the figure, the coefficient of friction between mass $m_1$ and the horizontal surface is 0.5.




a) What should the mass of object M be so that $m_1$ does not slide over the table?

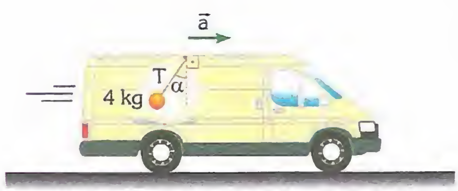
b) What would the acceleration of the system be if object M is removed?

**66.** Explain actual weight and apparent weight.

**67.** A 40-kg boy is standing on a scale in a lift, as shown in the figure. Find the apparent weight of the boy for the cases below:

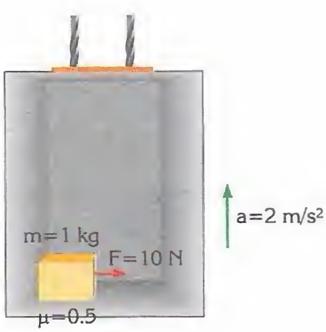
a) when the lift is stationary
b) when the lift is ascending at a constant velocity of 2 m/s
c) when the lift is descending at a constant velocity of 2 m/s
d) when the lift is ascending at an acceleration of 2 m/s$^2$
e) when the lift is descending at an acceleration of 2 m/s$^2$
f) when the ropes of the lift are broken

**68.**



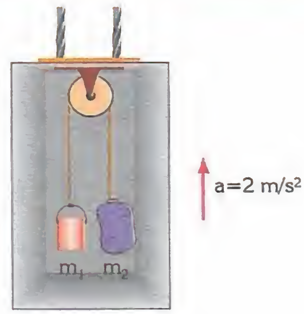

The van in the figure accelerates at 7.5 m/s$^2$. When the pendulum suspended from the roof of the vehicle comes to equilibrium, what will the tension in the string be?

**69.** If the elevator shown in the figure ascends with an acceleration of 2 m/s$^2$, find the acceleration of the object in the figure relative to

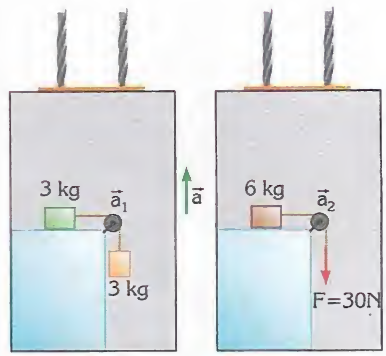



a) the elevator
b) the ground

**70.** The masses in a system in a lift, ascending with an acceleration of 2 m/s$^2$, as shown in the figure, are $m_1$=4 kg and $m_2$=6 kg. Find the accelerations of these masses according to an observer looking at the lift from outside.




**71.**




Both of the lifts in the figure are ascending with an acceleration of 2 m/s$^2$. Assuming the systems are frictionless, find the ratio of their accelerations $a_1/a_2$.

**72.** A parachutist has a mass of 60 kg and the mass of his parachute is 20 kg. If 80 N of average air resistance affects the parachutist after jumping out of the plane,


a) before opening his parachute, find the acceleration of the parachutist.
b) after opening his parachute, if the parachutist falls at a terminal velocity, find the air resistive force acting on him.

CHAPTER 5

# Torque and Equilibrum

*This chapter is about torque and the conditions for equilibrium. The torque concept helps explain how a small force can sometimes produce a large turning effect. These two conditions for equilibrium are essential for many engineering applications.*

## 5.1 FIRST CONDITION FOR EQUILIBRIUM

An object at rest (which continues to stay at rest) is said to be in **static equilibrium**. An object moving or rotating at a constant rate is in **dynamic equilibrium.**

For an object to be in equilibrium, two conditions must be fulfilled.

### First Condition for Equilibrium

For an object to be in equilibrium, the resultant force (vector sum of all forces) on it must be zero.

$$\vec{F}_{net} = 0$$

If n forces are acting on the object,

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ... + \vec{F}_n = 0$$

Accordingly the components of these forces must also add up to zero

$$\vec{F}_{1x} + \vec{F}_{2x} + \vec{F}_{3x} + ... + \vec{F}_{nx} = 0$$

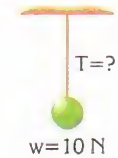
and

$$\vec{F}_{1y} + \vec{F}_{2y} + \vec{F}_{3y} + ... + \vec{F}_{ny} = 0$$

## Example 5.1

First condition for equilibrium

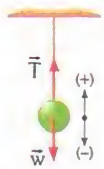
A 10 N lamp, suspended from the ceiling by a cord of negligible mass, is a rest, as shown in the figure. Find the tension in the cord.




### Solution

If the object stays at rest, the forces on it are in equilibrium.

$$\vec{F}_{net} = 0$$
$$\vec{T} + w = 0$$
$$T - w = 0$$
$$T = w$$
$$T = 10\ N$$

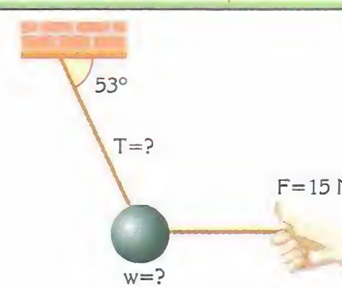



## Example 5.2

First condition for equilibrium

An object suspended from the ceiling is pulled by a horizontal force of 15 N, as shown in the figure. Find the tension in the string and the weight of the object.

(Take sin 53°=0.8 ; cos 53°=0.6)

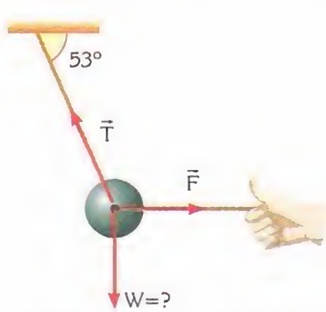



### Solution

There are three forces acting on the object in equilibrium. These forces are; the weight of the object ($\vec{w}$), the applied force ($\vec{F}$) and the tension ($\vec{T}$) in the string.

The forces acting on the object can be transferred to the xy coordinate system and then the first condition for equilibrium can be applied to the object on both the x and y axis.

$$\Sigma\vec{F}_x = 0$$
$$F - T_x = 0$$
$$T_x = F$$
$$T \cdot \cos 53° = 15\ N$$
$$T \cdot 0.6 = 15\ N$$
$$T = 25\ N$$




Using the first condition for equilibrium on the y axis, weight w can be obtained;

$$\Sigma\vec{F}_y = 0$$
$$T_y - w = 0$$
$$T_y = w$$
$$T \cdot \sin 53° = w$$
$$(25\ N)0.8 = w$$
$$w = 20\ N$$

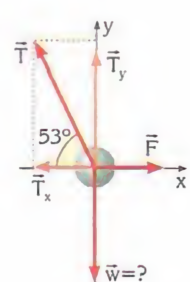



Figure 5.1 A spanner is used to tighten a screw

## 5.2 TORQUE

It is difficult to tighten a screw with bare hands. However, with a spanner the screw can be tightened easily, as shown in Figure 5.1. The spanner may produce a large turning effect.

The turning effect of a force is called the **torque** and is denoted by the Greek letter $\tau$. Torque is also called the **moment of a force.**

The point about which a force tends to rotate an object is called the **pivot (turning point).** In Figure 5.1 the screw acts as the pivot for the spanner.

The same force can produce different torques on an object, depending on the point of application and the orientation. Figure 5.2 illustrates this fact. In all four cases in Figure 5.2.a-d, a long rod is free to rotate around the pivot at point O.

The torque acting in Figure 5.2.a is greater than the torque acting on the rod in Figure 5.2.b, although the forces in both figures are equal in magnitude.

In Figure 5.2.a and 5.2.c, the forces act at the same point, however, the turning effects are different since the magnitudes of their forces are different.

The torque acting upon the rod in Figure 5.2.d is zero, although there is a non-zero force acting on the object.

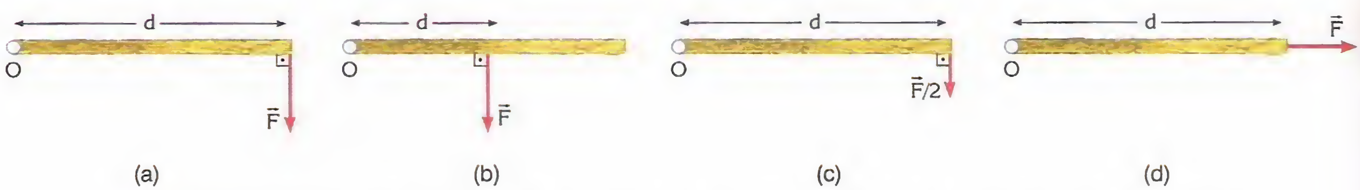



Figure 5.2 Of the four cases, torque in (a) is a maximum, torques in (b) and (c) are equal, torque in (d) is zero

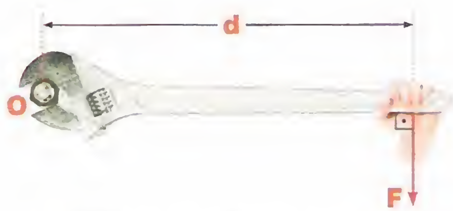



Figure 5.3 Torque produced by a force takes its maximum value when the force on an object is perpendicular to the line connecting the pivot to the application point of the force

Therefore, the turning effect (torque) produced by a force depends on three parameters:

a) Magnitude of the force (F)

b) The distance between the application point of the force and the pivot around which the object rotates (d)

c) The angle between the force vector and the line connecting the pivot to the application point of the force ($\theta$)

When a force on an object is **perpendicular** to the line connecting the pivot to the application point of the force, as shown in Figure 5.3, the torque produced by this force has its maximum value. In this case the torque is given by

$$\tau = Fd \quad (F \perp d)$$

When a force on an object is parallel to the line connecting the pivot to the application point of the force, as shown in Figure 5.4, the torque produced by this force is zero.

$$\tau = 0 \quad (F // d)$$

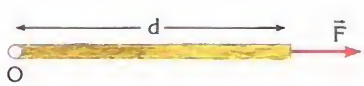



Figure 5.4 Torque is zero when the line of force passes through the pivot.

In general, a force can always be resolved into its components, one parallel, the other perpendicular to the line connecting the pivot to the application point of the force, as shown in Figure 5.5. In this case the torque produced by the force F equals the torque produced by its component perpendicular to the distance to the pivot. This is because the contribution of the parallel component of force to the torque is zero.

$$\tau = F_{\perp} d$$

$$\tau = (F \sin\theta)$$

$$\tau = F\, d \sin\theta$$

where θ is the angle between the force and the line connecting the pivot to the application point of the force.

The SI unit for torque is Nm.

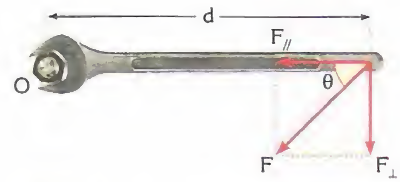
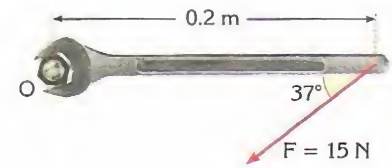



Figure 5.5 The components of a force parallel and perpendicular to the distance, d.

## Example 5.3

Torque

The force applied to a 0.2 m long spanner is 15 N, as shown in the figure. Find the torque produced by this force. (Take cos 37°=0.8; sin 37°=0.6)




### Solution

The component of force, $F_x$ does not cause any torque. Let us find its component along the y axis, $F_y$ which is perpendicular to the spanner. $F_y$ causes clockwise rotation.

$$F_y = F \sin\theta$$

$$F_y = (15\ \text{N})\,0.6 = 9\ \text{N}$$

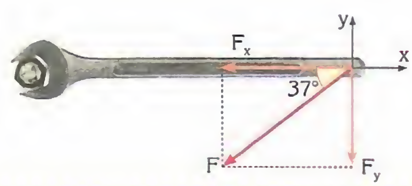
Its torque is,

$$\tau = F_y d$$

$$\tau = F d \sin 37$$

$$\tau = (9\ \text{N})(0.2\ \text{m})$$

$$\tau = 1.8\ \text{Nm}$$




## The Net Torque and the Direction of Rotation

When several forces act on a body, the torque caused by each force is calculated about the same turning point, then the sum of all the individual torques gives the net (resultant) torque. Consider the door in Figure 5.6, viewed from above. Two forces, $F_1$ and $F_2$, act on the door producing turning effects in opposite senses of rotation. Taking the anti-clockwise sense of rotation as positive and the clockwise sense of rotation as negative. The torques produced by the forces are

$$\tau_1 = F_1 d_1 \quad \text{and} \quad \tau_2 = -F_2 d_2$$

The net (resultant) torque on the door is

$$\tau_{net} = \tau_1 + \tau_2$$

$$\tau_{net} = F_1 d_1 - F_2 d_2$$

Now consider a stationary object:

If the net torque on the object is positive, then the body starts to rotate in the positive direction.

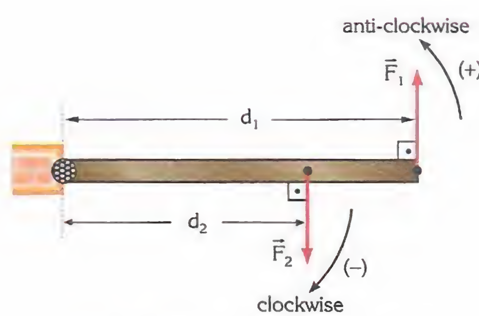



Figure 5.6 The top view of a door acted on by two forces. The net torque is the sum of the individual torques

If the net torque on the object is negative, then the body starts to rotate in the negative direction.

If the net torque due to the forces acting upon the object is zero, (the net clockwise torque equals the net anti-clockwise torque in magnitude) then the object does not start to rotate

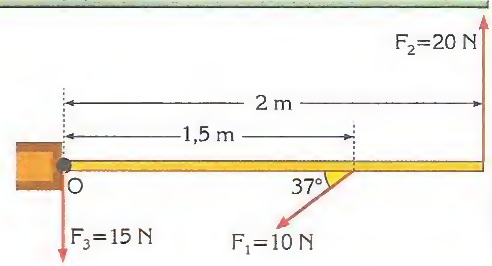
## Example 5.4

Net torque

Three forces are applied to a door, as shown in the figure.

a) Find the net torque acting on the door.

b) Calculate the minimum value of the fourth force which must be applied to prevent rotation of the door and determine its direction and point of application. (Take sin 37=0.6; cos 37=0.8)




### Solution

a) The vertical component of force $F_1$ produces a clockwise (–) rotation and force $F_2$ produces an anti-clockwise (+) rotation. Force $F_3$ doesn't produce any rotation because it is applied at the turning point.

$$\tau_1 = -F_{1y}d_1 = -F_1 d_1 \sin\theta \qquad \tau_3 = F_3 d_3$$

$$\tau_1 = -(10\text{ N})\,(1.5\text{m})0.6 \qquad \tau_3 = (15\text{ N})\,(0\text{ m})$$

$$\tau_1 = -9\text{ Nm} \qquad \tau_3 = 0$$

$$\tau_2 = F_2 d_2$$

$$\tau_2 = (20\text{ N})\,(2\text{ m})$$

$$\tau_2 = 40\text{ Nm}$$

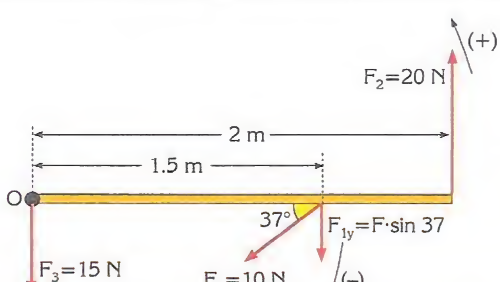
Thus the net torque about point O is;

$$\Sigma\tau_O = -\tau_1 + \tau_2$$

$$\Sigma\tau_O = -9\text{ Nm} + 40\text{ Nm}$$

$$\Sigma\tau_O = 31\text{ Nm}$$

Since the net torque is positive, the door rotates in the anti-clockwise direction with a magnitude of 31 Nm.



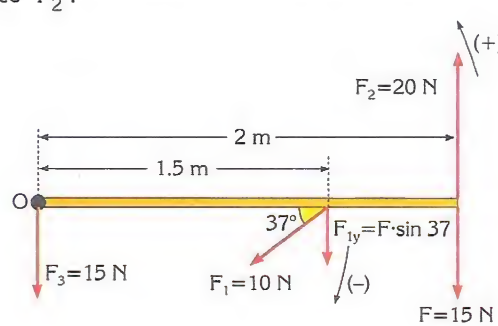

b) Since the net torque is anti-clockwise and its magnitude is 31 Nm. This means that, to prevent the door from rotating, a clockwise torque of 31 Nm must act on it. The minimum force to produce this torque must be applied perpendicularly, to the furthest point from the fulcrum. So, the force which must be applied 2 m away from the fulcrum is;

$$\tau_2 = Fd$$

$$31\text{ Nm} = F\,(2\text{m})$$

$$F_{min} = 15{,}5\text{ N}$$

This force must be applied in the opposite direction to force $\vec{F}_2$.




## 5.3 SECOND CONDITION FOR EQUILIBRIUM

An object may not be in equilibrium even though the first condition is fulfilled. Consider the object in Figure 5.7. The net force acting upon the object is zero, but the object does not remain at rest. Therefore for an object to be in equilibrium, one more condition is needed.

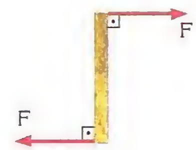



**Figure 5.7** Net force is zero but the object is not in equilibrium

Second condition for equilibrium

For an object to be in equilibrium, the resultant torque (sum of all torques) acting on it must be zero.

$$\tau_{net} = 0$$

If n forces produce torque on the object,

$$\tau_1 + \tau_2 + \tau_3 + ... + \tau_n = 0$$

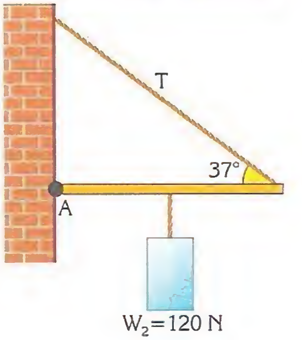
When $F_{net}=0$ but $\tau_{net} \neq 0$ on an object, the object is in **translational equilibrium**. The object does not accelerate, but starts to rotate.

When $F_{net} \neq 0$ but $\tau_{net} = 0$ on an object, the object is in **rotational equilibrium**. It does not start to rotate, but it accelerates.

## Example 5.5

Conditions for equilibrium

A weightless beam is hinged at point A and suspended by a rope at point B to a wall. A 120 N object is suspended from the midpoint. If the beam is in equilibrium, as shown in the figure find



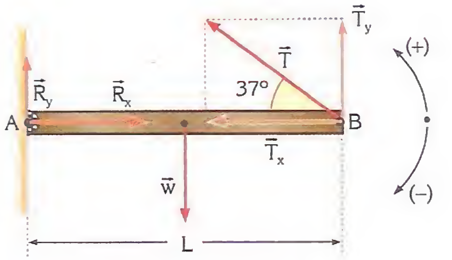

a) the tension (T) in the rope,

b) the reaction force (R) acting upon the beam exerted by the wall at point A.

(Take sin 37°=cos 53°=0.6; sin 53°=cos 37°=0.8)

**Solution**

a) Forces acting on the beam are as shown in the figure. $R_x$ and $R_y$ are the components of the force exerted by the wall on the beam. The torques produced by $R_x$ and $R_y$ are zero because they are exerted on the beam at the point of rotation. The net torque is produced by the forces T and w. Applying the second condition for equilibrium to the beam for the hinge at point A;




$$\Sigma\tau_A = 0$$

$$T_y L - w\left(\frac{L}{2}\right) = 0$$

$$(T\sin 37°)L - 120\left(\frac{L}{2}\right) = 0$$

$$T = 100 \text{ N}$$

b) From the first condition for equilibrium;

$$\Sigma F_x = 0$$

$$R_x - T_x = 0$$

$$R_x - 100 \cdot \cos 37° = 0$$

$$R_x = 80\ \text{N}$$

$$\Sigma F_y = 0$$

$$R_y + T_y - w = 0$$

$$R_y + 100 \cdot \sin 37° - 120 = 0$$

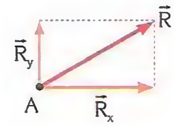
$$R_y = 60\ \text{N}$$

Thus, the reaction force can be calculated using Pythagorean theorem;

$$R^2 = R_x^2 + R_y^2$$

$$R^2 = 80^2 + 60^2$$

$$R = 100\ \text{N}$$



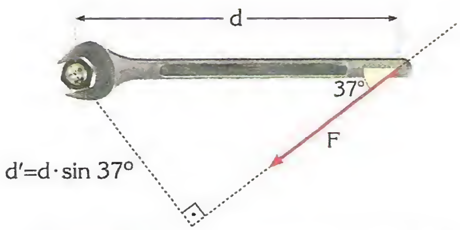




Figure 5.8 A distance perpendicular to the line of action of the force.

## A Different Interpretation of the Torque Formula

The torque formula

$$\tau = Fd\sin\theta$$

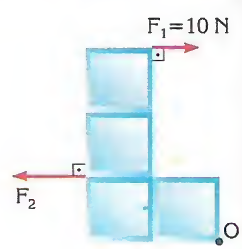
can be interpreted as

$$\tau = F(d\sin\theta)$$

Examining Figure 5.8 shows that ($d\sin\theta$) is the perpendicular distance from the pivot to the line of action of the force.

## Example 5.6 — Second condition for equilibrium

The box of negligible weight in the figure is free to rotate about point O. The box is subject to two forces $F_1$ and $F_2$, as shown in the figure. If $F_1$ is 10 N and each side of the square is 1 m, find the force $F_2$ required for the box to be in equilibrium.




### Solution

For equilibrium

$$\Sigma\tau = 0$$

$$F_2 d_2 - F_1 d_1 = 0$$

$$F_2(1\ \text{m}) - (10\ \text{N})(3\ \text{m}) = 0$$

$$F_2 = 30\ \text{N}$$

Therefore either of the two interpretations of the torque formula can be used

Torque = (Perpendicular force)×(Distance)

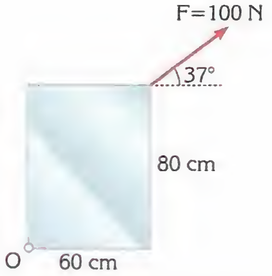
Torque = (Force)×(Perpendicular distance)

In the most general case, the components of both the force and the distance can be used to find the component torques, then, these can be added as shown in Example 5.7.

## Example 5.7

Torque

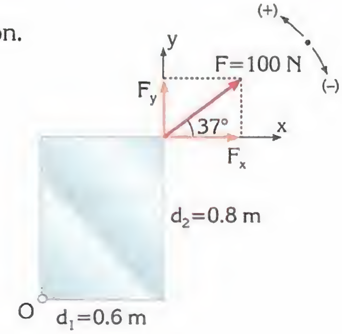
Find the torque on the object around the pivot at point O. ($\sin 37° = 0{,}6$; $\cos 37° = 0{,}8$)




**Solution**

$F_x$ produces (–) rotation and $F_y$ produces (+) rotation.
Thus the net torque about point O is;

$$\begin{aligned}\Sigma\tau &= -F_x \cdot d_1 + -F_y \cdot d_2 \\ &= -(F\cdot\cos 37°)\cdot d_1 + (F\cdot\sin 37°)\cdot d_2 \\ &= -100\cdot 0{,}8\cdot 0{,}8 + 100\cdot 0{,}6\cdot 0{,}6 \\ &= -64 + 36 \\ &= -28 \text{ N.m} \quad \text{(Net torque is in (–) direction.)}\end{aligned}$$



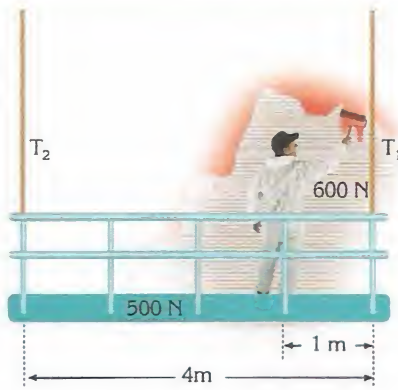

### Choosing the Pivot

If an object is in equilibrium, the total torque with respect to any point is zero. Thus a pivot point at any place suitable for the problem can be chosen.

## Example 5.8

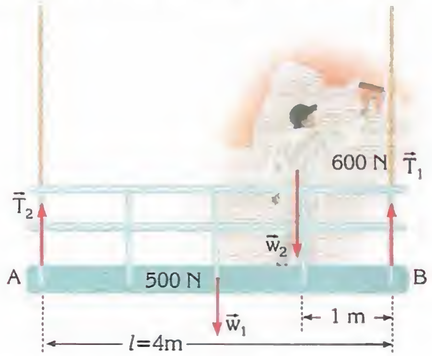
Addition of torques

A painter weighing 600 N is painting a wall while standing on a beam weighing 500 N, suspended by ropes. While the painter is at the position shown in the figure, what are the tensions $T_1$ and $T_2$ in the ropes? (Other weights in the system are negligible)




**Solution**

First sketch the forces acting on the system.




Since the system is in equilibrium, the equilibrium of torques can be used. By selecting first the application point of $T_2$, then the application point of $T_1$, first $T_1$ and then $T_2$ can be calculated.

Hence, since;

$$\Sigma \vec{\tau}_A = 0$$

$$T_2\cdot 0 + T_1\cdot l - 500\cdot\frac{l}{2} - 600\left(l - \frac{l}{4}\right) = 0$$

$$T_2\cdot 0 + T_1 4 - 500\cdot\frac{4}{2} - 600\left(4 - \frac{4}{4}\right) = 0$$

$$T_1 = 700 \text{ N}$$

And since;

$$\Sigma \vec{\tau}_B = 0$$

$$T_1 \cdot 0 + 500 \cdot \frac{l}{2} + 600\frac{l}{4} - T_2 \cdot l = 0$$

$$T_1 \cdot 0 + 500 \cdot \frac{4}{2} + 600\frac{4}{4} - T_2 4 = 0$$

$$T_2 = 400 \text{ N}$$

Note that the equilibrium of forces ($\Sigma\vec{F}_y=0$) cannot be used to calculate $T_1$ and $T_2$. For these types of problems, the equilibrium condition for torques is generally used for calculating the unknown values. However, if the distance l/4 is given to be x, as the number of unknown values would be greater, we would need to derive further equations. In this case, we would use $\Sigma\vec{F}_y=0$ as well as obtaining the torque about different points.

## 5.4 CENTRE OF GRAVITY

### a. Location of the Resultant Force

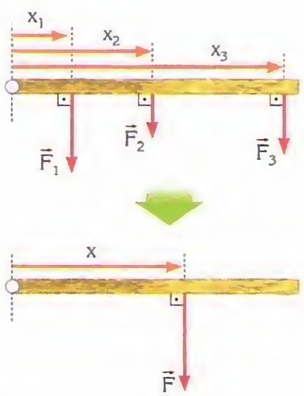



**Figure 5.9** The resultant of parallel forces

Consider the weightless rod acted on by three parallel forces at different points, as shown in Figure 5.9. If the three forces are to be replaced by a single force such that the net force and the net torque on the object, with respect to the given pivot, remains constant, what is the magnitude and point of application of this single resultant force?

Since the net force must remain constant

$$F = F_1 + F_2 + F_3$$

The distance from the pivot can be found by using the equality of torques

$$\tau = \tau_1 + \tau_2 + \tau_3$$

$$(F_1 + F_2 + F_3)x = F_1x_1 + F_2x_2 + F_3x_3$$

$$x = \frac{F_1x_1 + F_2x_2 + F_3x_3}{F_1 + F_2 + F_3}$$

### b. The Centre of Gravity

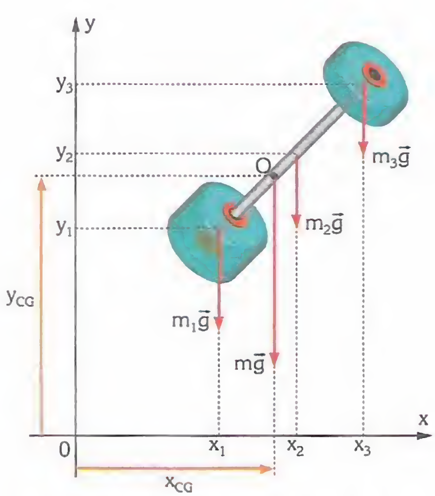



**Figure 5.10** Centre of gravity of an object

Since gravitational force always acts towards the centre of the Earth, gravitational forces affect all the particles exerting forces upon the particles in the same direction parallel to each other, as shown in Figure 5.10. The resultant of these forces produces the weight of the body. The application point of this resultant force is called the **centre of gravity** of the body.

A barbell with different weights on each end is placed in the xy coordinate system in Figure 5.10. The barbell can be considered to be composed of three pieces (Two weight disks and the bar). The weights of these parts are denoted to be $m_1\vec{g}$, $m_2\vec{g}$ and $m_3\vec{g}$. Thus, the total weight of the barbell is;

$$m\vec{g} = m_1\vec{g} + m_2\vec{g} + m_3\vec{g}$$

The application point of $m\vec{g}$ is point O.

Writing the torque equation as;

$$(m_1g+m_2g+m_3g)x_{CG}=m_1gx_1+m_2gx_2+m_3gx_3$$

where $(m_1+m_2+m_3)$ is the total mass of the object. This equation can be rearranged as;

$$x_{CG}=\frac{(m_1x_1+m_2x_2+m_3x_3)g}{(m_1+m_2+m_3)g}$$

Simplifying g;

$$x_{CG}=\frac{\Sigma(mx)}{\Sigma m}$$

For the y axis;

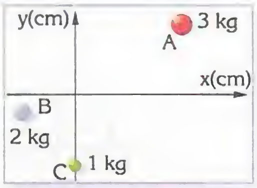
$$y_{CG}=\frac{(m_1y_1+m_2y_2+m_3y_3)g}{(m_1+m_2+m_3)g} \qquad y_{CG}=\frac{\Sigma(my)}{\Sigma m}$$

$x_{CM}$ and $y_{CM}$ are the coordinates of the body's centre of mass. The centre of mass is a point where the whole mass of a body is assumed to be concentrated. In an environment where the gravitational field is uniform, the centre of gravity and mass are at the same point. In gravity-free environments, as there is no weight, there is only the centre of mass.

## Example 5.9

Centre of mass

Objects A, B and C are shown in the xy coordinate system. If the masses of the objects are 3 kg, 2 kg and 1 kg, respectively, find the coordinates of the centre of mass of the system formed by these objects.




### Solution

The masses of the objects and the coordinates of their positions are

3 kg, A(6 cm; 4 cm)

2 kg, B(–3 cm; –1 cm)

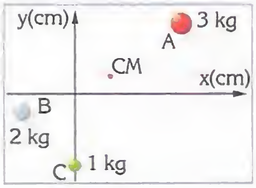
1 kg, C(0 cm; –4 cm)

The coordinates of the centre of mass are

$$x_{CM}=\frac{\Sigma(m\cdot x)}{\Sigma m}=\frac{m_Ax_A+m_Bx_B+m_Cx_C}{m_A+m_B+m_C}$$

$$=\frac{(3\text{ kg})(6\text{ cm})+(2\text{ kg})(-3\text{ cm})+(1\text{ kg})(0\text{ cm})}{3\text{ kg}+2\text{ kg}+1\text{ kg}}$$

$$x_{CM}=2\text{ cm}$$




$$y_{CM}=\frac{\Sigma(m\cdot y)}{\Sigma m}=\frac{m_Ay_A+m_By_B+m_Cy_C}{m_A+m_B+m_C}$$

$$=\frac{(3\text{ kg})(4\text{ cm})+(2\text{ kg})(-1\text{ cm})+(1\text{ kg})(-4\text{ cm})}{3\text{ kg}+2\text{ kg}+1\text{ kg}}$$

$$y_{CM}=1\text{ cm}$$

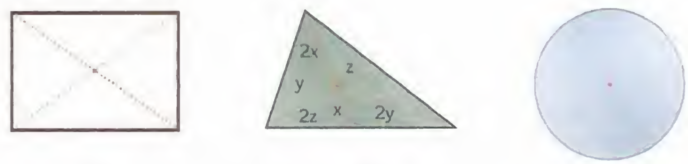
Thus, the coordinates of the centre of mass are 2 cm and 1 cm.

## 1) Centre of Gravity of Some Geometrical Shapes

For the rectangular frame shown in Figure 5.11.a, the centre of mass is at the intersection point of its diagonals.

For the triangular plate shown in Figure 5.11.b, the centre of mass is at the intersection point of the medians.The intersection of the medians is at 2/3 of the distance from the vertex. (The centre of mass of a triangular frame cannot be found by this method.)

For the circular plate shown in Figure 5.11.c, the centre of mass is at its geometrical centre.




(a) (b) (c)

Figure 5.11 a) Rectangular frame, b) triangular plate, c) circular plate.

Figure 5.12 In order to find the centre of gravity of an object, it must be suspended from various points. The intersection points of the extension lines through these points is the centre of gravity of the object.

## 2) Finding the Centre of Gravity of an Irregular Object

Whenever an object is suspended freely, the centre of gravity is always on the vertical line along which it is suspended. The centre of gravity of bodies can be found using this fact.

In Figure 5.12, the object is suspended from a point and a vertical line is

drawn in. When it is hung from another point and a similar vertical line is drawn in, the intersection point

shows the centre of gravity.

## 3) States of Equilibrium

There are three types of equilibrium states. These are stable, unstable, and neutral equilibrium. The cone shown in Figure 5.13.a is in a state of stable equilibrium. The forces on the cone are balanced. The centre of gravity is over the base. If the cone is pushed a little at the top, it will return to its original position.

Its wide base and lower centre of gravity makes the cone more stable. In Figure 5.13.b, the top of the cone is a tiny point. A small push will cause it to lose its equilibrium position and small force applied to the cone will cause a torque, and the cone will fall. In Figure 5.13.c, rolling the cone will not cause any change in the line between its centre of gravity and its base. Wherever it lies, its centre of gravity will be towards the contact area.

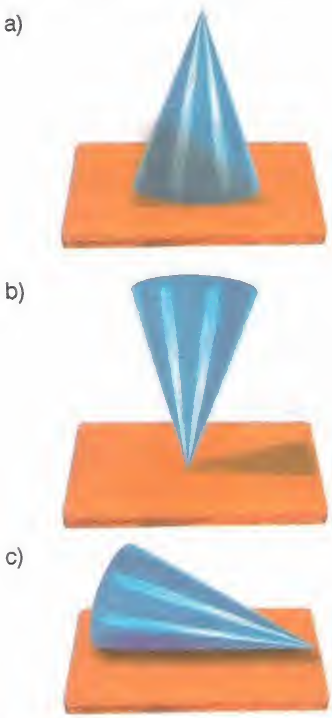
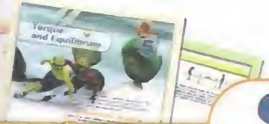
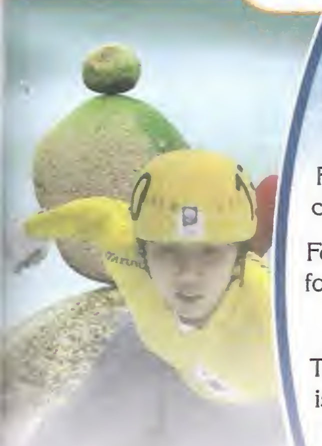



Figure 5.13 a) Stable equilibrium, b) Unstable equilibrium, c) Neutral equilibrium



# Summary



An object which is at rest (and remains at rest) is said to be in static equilibrium. An object moving or rotating at a constant rate is in dynamic equilibrium.

For an object to be in equilibrium, two conditions must be fulfilled.

For an object to be in equilibrium, the resultant force (vector sum of all forces) on it must be zero.

$$\vec{F}_{net} = 0$$

The turning effect of a force is called **torque** and is denoted by the Greek letter $\tau$. Torque is also called the moment of a force.

$$\tau = Fd\sin\theta$$

Torque can act in either a clockwise or anti-clockwise direction.

An object may not be in equilibrium even though the first condition is fulfilled.

The net force on an object may be zero but the object may not remain at rest. Therefore for an object to be in equilibrium, one more condition is needed.

For an object to be in equilibrium, the resultant torque (sum of all torques) acting on it must be zero.

$$\tau_{net} = 0$$

The coordinates of the centre of gravity of an object in the xy coordinate system are obtained from the following equations;

$$x_{CG} = \frac{\sum(mx)g}{\sum mg}$$

And

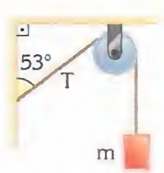
$$y_{CG} = \frac{\sum(my)g}{\sum mg}$$

# QUESTIONS AND PROBLEMS

### 5.1 First Condition for Equilibrium

**1.** What is the first condition for equilibrium?

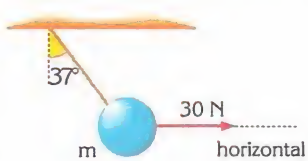
**2.** The tension in the rope is 30 N when an object of mass m is in equilibrium, as shown in the figure.




What is mass, m, in kilograms?

(Take sin 53°=0.8; cos 53°=0.6; g=10 N/kg)

**3.** An object of mass m is in equilibrium with the aid of a horizontal force of 30 N, as shown in the figure.

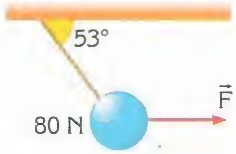



What is the mass of the object, in kilograms?

(Take sin 37°=0.6; cos 37°=0.8; g=10 N/kg)



4. An 80 N object suspended from a ceiling is pulled by a horizontal force of F, as shown in the figure.



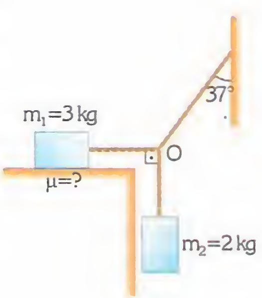

a) Find the tension in the string

b) Find the force F (Take cos 53°=0,6, sin 53°=0,8)

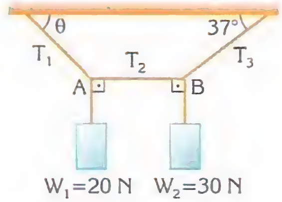
5. The spherical object shown in the figure is in equilibrium.


What is the tension force in the string, in Newtons, if the reaction force applied to the object by the inclined plane is 30 N? (Take sin 53°=0.8; cos 53°=0.6)

6. The system in the figure is at rest. If the smallest object which is placed on mass $m_2$ can cause the system to move, calculate the maximum value of the coefficient of static friction between the mass $m_1$ and the surface.

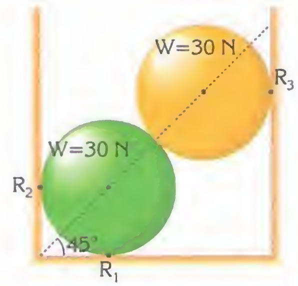



7. Two objects having weights $W_1$=20 N and $W_2$=30 N, are suspended from a ceiling by different strings and are brought into equilibrium, as shown in the figure. If the string AB is horizontal, find the tensions $T_1$, $T_2$, $T_3$ and calculate the value of the angle θ.




(sin 37°=cos 53°=0.6 ; sin 53°=cos 37°=0.8)

8. Two identical spheres having equal weights of 30 N are placed in a rectangular container, as shown in the figure. The system is in equilibrium and the centres of the spheres are in line with the corner of the wall and make an angle of 45° with the floor. Find the force, T, which the spheres exert on each other and the reaction forces $R_1$, $R_2$, $R_3$ which are exerted on the spheres by the walls. (Neglect friction.)

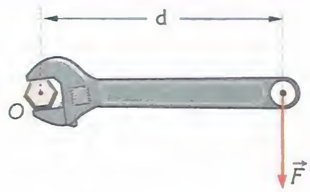



## 5.2 Torque

9. What is torque?

10. What does torque depend upon?

11.




What is the torque produced by a 5 N force, perpendicular to the 2 m spanner, as shown in the figure above?

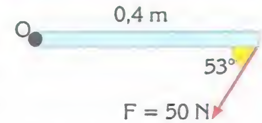
12. A driver applies a force F to the steering wheel of his car with each hand, as shown in the figure. If the wheel has a radius d, find


a) the net force

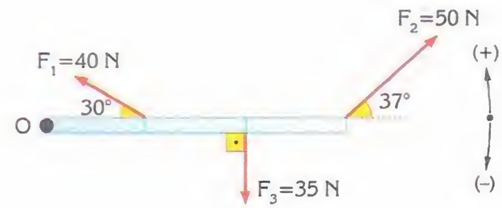
b) the net torque acting on the wheel.

13. A child applies a force of 450 N to the pedal of his bicycle in a downward direction when the arm of the pedal makes an angle of 37° with the horizontal, as shown in the figure. If the arm of the pedal has a length of 0.2 m, what is the torque produced by the force?


14. A force of 50 N is applied to a 0.4 m road that can turn about point 0 in the horizontal plane, as shown in the figure. Find the torque produced by this force. (Take sin53°=0.8)




15.



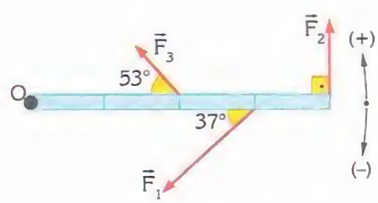

Three forces $F_1$ = 40 N, $F_2$ = 50 N, and $F_3$ = 35N, are acting on an equally divided 3 m rod which can turn about point 0 in the horizontal plane, as shown in the figure.

a) find the torque produced by each force.

b) find the total torque.

(Take sin 37° = 0.6, sin 30° = 0.5)

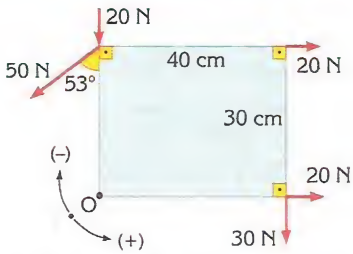
16.




The forces $F_1$ = 20N, $F_2$ = 10N and $F_3$ = 5N are acting on an equally divided 2 m rod which can turn about point 0 in the horizontal plane, as shown in the figure. Find the resultant torque.

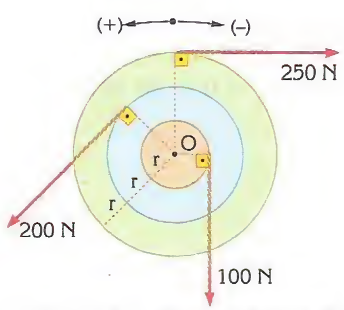
(Take sin 37° = 0,6; sin 53° = 0,8)

17.




The forces 20 N, 30 N, and 50 N are applied to a square plate which can rotate about point 0 in the horizontal plane, as shown in the figure. Find the total torque acting on the plate. (Take sin 53° = 0.8)

18.

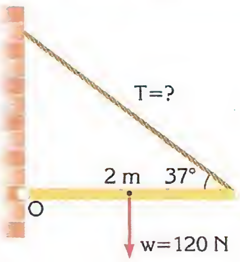



Three discs with radii r, 2r, and 3r are fixed to each other and can all turn about point 0. They are rotated by ropes exerting forces of 100 N, 200 N, and 250 N, as shown in the figure. If the radius r=0.1m, what is the resultant torque acting on this system?

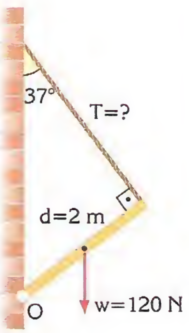
### 5.3 Second Condition for Equilibrium

19. What is the second condition for equilibrium?

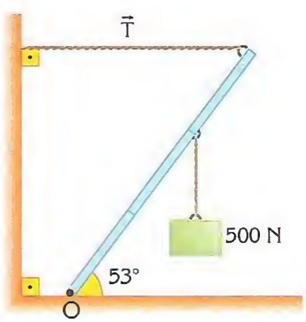
20. A 2 m long rod which is suspended by a rope is at rest, as shown in the figure. If its weight is 120 N, what is the tension T in the rope? (The weight of the rod is at its geometric centre.)




21. A rod of weight 100 N and a length of 2 m is hung as shown in the figure. If it is in equilibrium, find the tension in the rope assuming that the weight of the rod is at the geometric centre of the rod.




22.



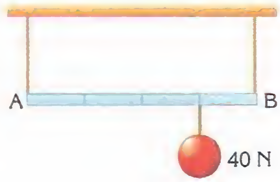

A 3 m rod of negligible mass pivoted at point 0 is tied to the wall with a rope, as shown in the figure. If a 500 N load is suspended 2 m from point 0

a) find the tension T in the rope.

b) find the horizontal and vertical forces that the rod experiences at point 0.

(Take sin 53° = 0,8; sin 37° = 0,6)

**23.** A beam of negligible mass is suspended from both of its ends. If it has a length of 4m divided into equal sections and a 40N sphere is suspended from it, as shown in the figure, find the tensions in the ropes.

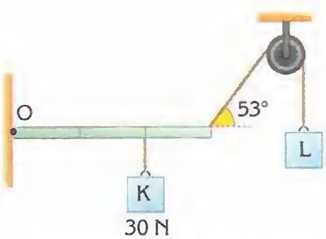



**24.** One end of a beam of negligible mass is hinged to a support and the other end is suspended from the ceiling with a string. If a 60 N weight is located on the beam, as shown in the figure


a) find the tension in the string

b) find the force applied by the support to the beam

**25.**



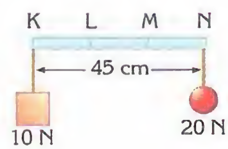

An equally divided rod of negligible mass is in equilibrium with objects K and L, as shown in the figure. If the weight of the object K is 30 N, find

a) the weight of object L

b) the forces applied by the wall to the rod
(Take cos53° = 0.6; sin53° = 0.8)

**26.** A uniform rod weighing 10 N is suspended by two ropes, as shown in the figure. What is the ratio of the tensions in the ropes $T_1 / T_2$?


**27.** Two objects weighing 10 N and 20 N are attached to the ends of a 45 cm long rod which is equally divided, as shown in the figure. Neglecting the weight of the rod




a) for the rod to remain in equilibrium in the horizontal direction, how many cm away from point K should the rod be suspended?

a) what is the tension in the string used to suspend the rod?

**28.**


Objects weighing P, 2P and 3P are suspended from an equally divided rod, as shown in the figure. Neglecting the weight of the rod

a) for the rod to remain in equilibrium in the horizontal direction, from which point should the rod be suspended?

b) What is the tension in the string from which the rod is suspended, in terms of P?

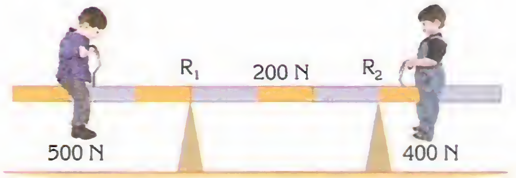
**29.** A uniform sphere of weight 28 N is attached to a frictionless wall by a string, as shown in the figure. If the sphere is in equilibrium and the extension of the string passes through the centre of the ball, calculate


a) the tension in the string

b) the reaction force of the wall

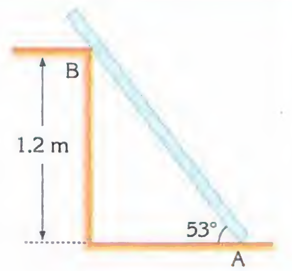
(Take $\sin 37° = 0.6$; $\cos 37° = 0.8$)

**30.**




A uniform beam of weight 200 N having 8 equal divisions is placed on two supports and is in equilibrium when two children sit on each end, as shown in the figure. What is the ratio of the reaction forces applied to the beam by the supports ($R_1 / R_2$)?

**31.** A uniform 2 m long wooden beam with a weight of 140 N, leans against a 1.2 m high wall, as shown in the figure. There is no friction at point B where the beam touches the wall. If the beam is in equilibrium, calculate

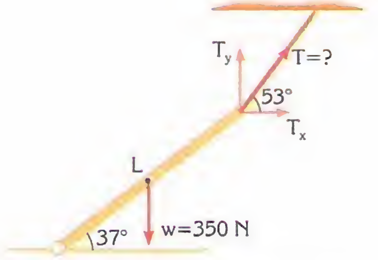



a) the reaction force ($N_B$) of the wall applied to the beam at point B

b) the normal force ($N_A$) which the ground exerts on the beam at point A

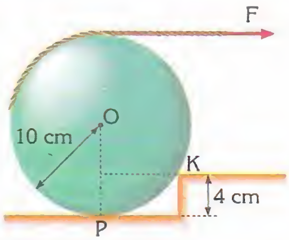
c) the friction force (f) which the ground exerts on the beam at point A.

**32.**




A 350N rod hung by a rope is in equilibrium, as shown in the figure. What is the tension in the rope (Take $\sin 37°=0{,}6$; $\cos 37°=0{,}8$, $\sin 53°=0{,}8$; $\cos 53°=0{,}6$)

33. An attempt is made to raise a uniform sphere of weight 30 N and radius 10 cm onto a 4 cm high step using a horizontal force F. The force is applied to a string wrapped around the sphere, as shown in the figure. Find

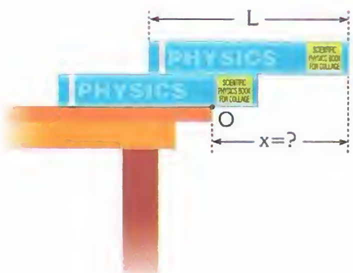



a) the minimum value of F

b) the reaction force at point K.

(the weight of the string is negligible)

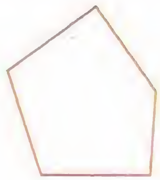
34.




Two identical, uniform books of length L, having equal weights, are placed on a table one above the other, as shown in the figure. For the books to remain in equilibrium in this position, how far away from point O can the outer edge of the upper book be placed?

## 5.4 Centre of Gravity

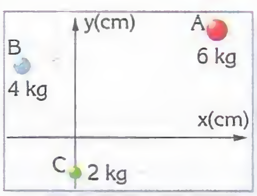
35. How do you find the centre of gravity of an irregular shaped piece of paper, as shown in the figure?



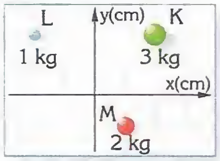
36. Where is centre of mass of a square?

37. Where is the centre of mass of a triangle?

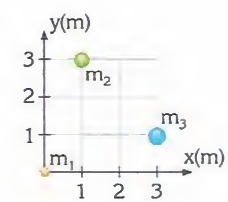
38. Objects A, B, and C of masses 6 kg, 4 kg, and 2 kg, respectively are placed in the xy coordinate system, as shown in the figure. Assuming each side of the squares is 1cm long, find the coordinates of the centre of mass of the system formed by these objects.



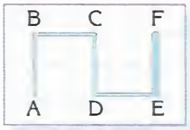
**39.** Homogeneous spheres K, L and M are shown in the xy coordinate system. If each side of the squares is 1cm long, find the coordinates of the centre of mass of the system formed by the spheres.



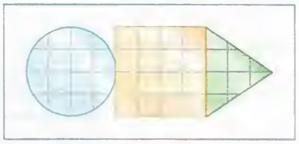

**40.** In the system shown in the figure; $m_1=1$ kg, $m_2=2$ kg and $m_3=3$ kg. What are the coordinates of the centre of mass of the system?




**41.** The homogeneous cable is bent twice between points E and F, as shown in the figure. If each side of the squares is 1cm long, find the coordinates of the centre of mass of the cable.

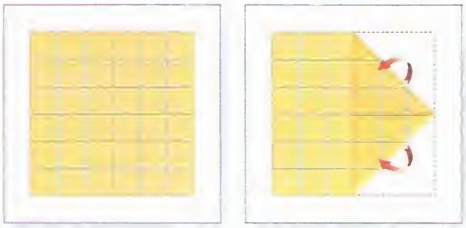



**42.**



Three uniform objects of the same thickness are attached to each other, as shown in the figure. If each side of the small squares is 1 cm, how many cm is the centre of mass of the system away from the centre of mass of the circular object? (Take $\pi=3$)

**43.**

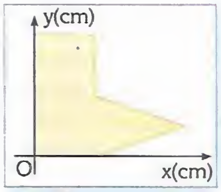


Şekil-I Şekil-II

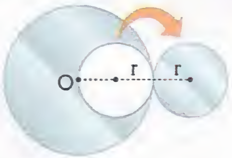
A square plate of side length 6 cm is folded up, as shown in figure II. By how many cm does the centre of mass of the system shift?

**44.** Find the centre of mass of the uniform plate shown in the figure.

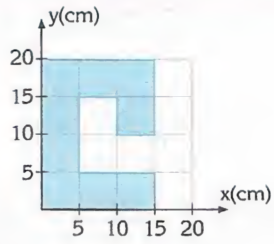
(The thickness of the plate is uniform.)



**45.** A circular section from a uniform circular plate having a radius of 2r is removed and attached to its rim, as shown in the figure. How far is the centre of mass of the new object from point O?




**48.** What are the coordinates of the centre of mass of a uniform piece of cardboard which is cut into the shape shown in the figure?

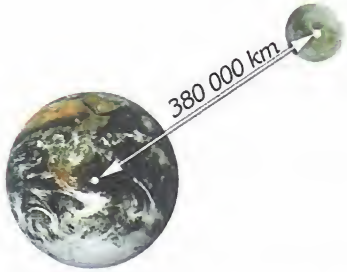



**46.**


From where should 6 uniform objects, which are attached to each other, as shown in the figure, be hung in order for the system to be in equilibrium in the horizontal direction?

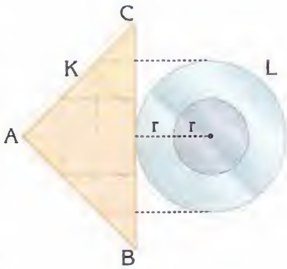
**49.** The radius of the circular ends of the empty cylindrical tin shown in the figure is r and its height is 2r. When one of the circular lids is cut off, how does its centre of gravity change?

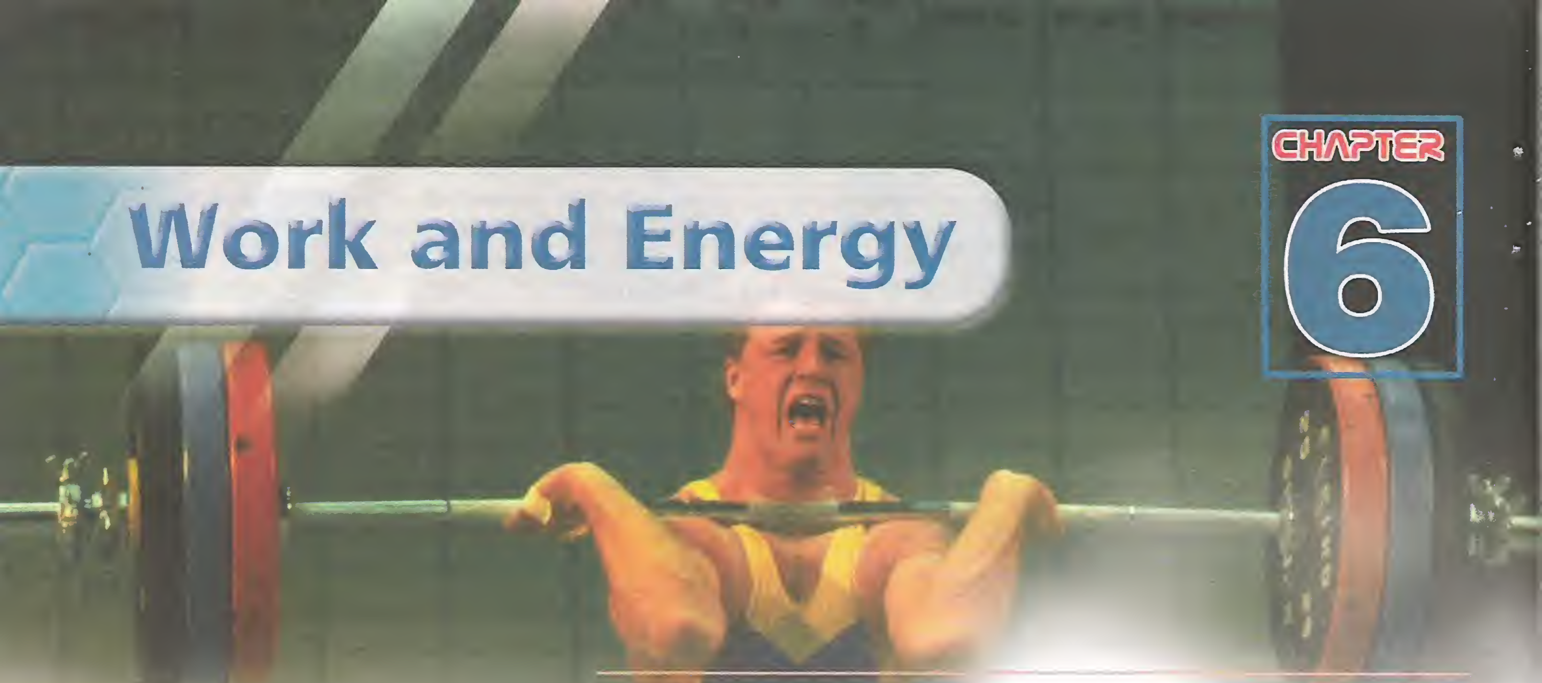

**47.**




Assuming that the mass of the Moon is $100^{th}$ of that of the Earth and the distance between them is 380 000 km, calculate the distance of the centre of mass of the Earth-Moon system from the centre of the Earth.

**50.** Uniform plates K and L, whose weights are equal, are joined, as shown in the figure. When the centre piece of radius r is removed from plate L, what will the difference be between the position of the new centre of gravity and the previous one?






# Work and Energy

*The weight lifter in the picture does a lot of work in raising the weight from the floor to its present position. However, he does no work while holding it steady above his head. Work done by the weight lifter in lifting the bar-bell is stored as the potential energy of the bar-bell. This potential energy turns into kinetic energy when he allows it to drop. In this chapter work in physics will be discussed, types of energy and the relationships between them will also be discussed.*

## 6.1 WORK

In our daily lives, work has a broad meaning. Work might be studying physics, driving a car, or digging a garden. However, in physics, work has a precise meaning. Work is done whenever a force causes an object to move.

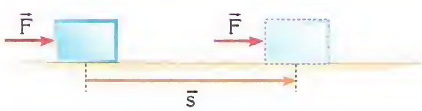



**Figure 6.1** Work done by a constant force

**Work** is defined as the product of a force and the distance through which it moves a body in the direction of that force. Therefore when the force and the displacement are parallel, as in Figure 6.1, work is given by

$$W = Fs \quad \text{(force and displacement parallel)}$$

Work is a scalar quantity. It does not have a direction associated with it. The SI unit for work is the joule (J).

$$\text{(Joule)} = \text{(Newton)(metre)}.$$

$$1\ \text{J} = 1\ \text{Nm}$$

In words, 1 joule is the work done by a force of 1 newton, when it acts on an object which moves in the direction of the force over a distance of 1 metre.

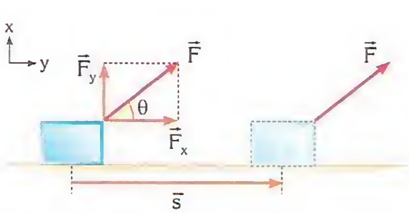



**Figure 6.2** Work done by the parallel component of a force

In general, when force and displacement are not parallel, it is always possible to resolve the force into two components, one parallel and one perpendicular to the displacement, as shown in Figure 6.2.

Let us denote the parallel component of force by $F_{//}$ and the perpendicular component by $F_{\perp}$.

Work done by the component of the force perpendicular to the direction of displacement is zero, because the object is not displaced in the direction of this component. Therefore, work done by a force equals the work done by its component parallel to the direction of the displacement.

$$W = F_{//}s$$

$$W = (F\cos\theta)s$$

$$W = Fs\cos\theta$$

Here θ is the angle between the force vector F and the displacement vector s. In conclusion, there are two conditions for an applied force to be capable of doing work in a physical sense:

- The object must be displaced. If the object does not move under the action of the applied force, as shown in Figure 6.3, the force does not do any work.
- The applied force or a component of the force must be parallel to the displacement.

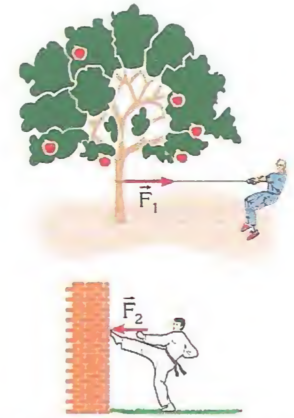



**Figure 6.3** Work is zero in both situations, because the objects do not move.

**Figure 6.4** The man doesn't do any work on the suitcase, because the force applied by his hand is perpendicular to the direction of the suitcase's displacement.

## Net Work

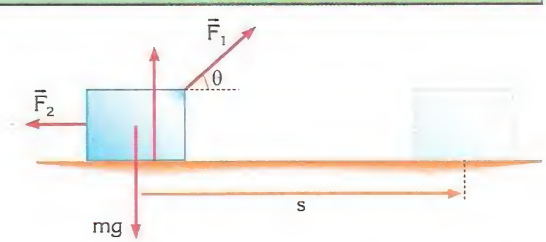
An object may move under the action of two or more forces. In this case the net work on the object is the algebraic sum of the work done by each force.

$$W_{net} = W_1 + W_2 + W_3 + ...$$

## Example 6.1

Calculating net work done

The object in the figure to the right moves under the action of four forces on a smooth surface, as shown in the figure. What is the net work done on the object when it moves a distance of 5m, if $F_1$=20 N, $F_2$=13 N and θ=37°? (Take cos 37° = 0.8)




### Solution

Work done by gravity and the normal contact force on the object are both zero because both forces are perpendicular to the displacement vector.

Work done by $F_1$ is,

$$W_1 = F_1 s\cos\theta$$

The force $F_2$ is parallel to the displacement, but points in the opposite direction. Therefore, the angle between the force $F_2$ and the displacement vector is 180°. Since cos180°=–1, work done by $F_2$ is,

$$W_2 = F_2 s\cos 180°$$

$$W_2 = -F_2 s$$

In general when force and displacement are parallel but oppositely directed, work is negative.

The net work done on the object along distance s is,

$$W_{net} = W_1 + W_2 + W_{mg} + W_N$$

$$W_{net} = F_1 s\cos - F_2 s + 0 + 0$$

Inserting the numeric values,

$$W_{net} = (20\ N)(5\ m)(\cos 37°) - (13\ N)(5\ m)$$

$$W_{net} = 80\ J - 65\ J$$

$$W_{net} = 15\ J$$

Note that the net work done on the object equals the work done by the net force on the object.

## Force - Position (F - x) Graphs

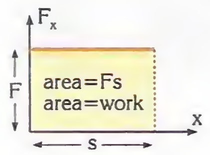



Figure 6.5 Area under the F-x graph equals work

Assume that a constant force F is applied to an object in the same direction as its displacement, which is taken to be along the x-axis. The force versus the object's position graph is as shown in Figure 6.5. The area under this graph is given by

$$\text{Area} = \text{Force} \times \text{distance}$$

Therefore, if force and displacement lie along the same axis, the area under the force-displacement graph equals work done by that force.

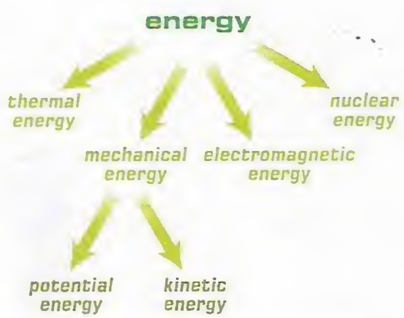



Figure 6.6 Various forms of energy

# 6.2 KINETIC ENERGY AND THE WORK-KINETIC ENERGY THEOREM

In relaxed terms, energy can be defined as "the capacity of a physical system to do work". Actually the energy concept is too complex to be defined in a simple short sentence. The concept of energy can be understood by examples of energy transfer and conservation of energy.

Some of the various types of energy are shown in Figure 6.6. This book mainly deals with mechanical and thermal (heat) energy.

Energy is a scalar quantity, it does not have a direction.

Work and energy have the same physical dimension. The SI unit of energy is the joule.

## a. Kinetic Energy

**Kinetic energy** is the energy an object has by virtue of its motion. An object at rest does not possess kinetic energy.

The kinetic energy of an object of mass m moving at a velocity v is given by

$$KE = \frac{1}{2}mv^2$$

In the formula mass is in kg and velocity is in m/s.

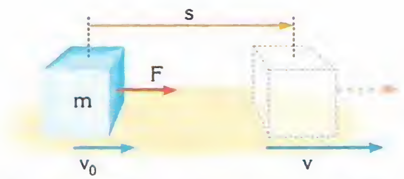



Figure 6.7 The kinetic energy of the object increases as positive work is done on it

## b. Relationship between Work and Kinetic Energy

Consider an object moving on a smooth, level surface, as shown in Figure 6.7. A force of constant magnitude acts on the object in the same direction as the motion of the object. The velocity of the object increases from an initial value of $v_0$ to a final value of v, as the object travels a distance, s, under the action of this force. Since the net force on the object is F, the net work done on the object along this distance is given by

$$W_{net} = Fs$$

Using the second law of motion,

$$F = ma$$

and the kinematical equation,

$$a = \frac{v^2 - v_0^2}{2s}$$

the expression for work can be written as,

$$W_{net} = Fs$$

$$W_{net} = (ma)s$$

$$W_{net} = m\left(\frac{v^2 - v_0^2}{2s}\right)s$$

$$W_{net} = m\frac{v^2 - v_0^2}{2s}$$

Finally,

$$W_{net} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

$\frac{1}{2}mv^2$ is defined as the kinetic energy of an object. Thus

$$W_{net} = KE_f - KE_i$$

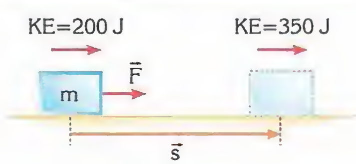
$$W_{net} = \Delta KE$$

In words, doing positive work on an object increases the kinetic energy of the object. When negative work is done on an object, the kinetic energy of the object decreases. The net work done on an object equals the change in kinetic energy of the object.

This expression is known as the **work-kinetic energy** theorem.

## Example 6.2 Determining work done from the change in kinetic energy

The kinetic energy of an object increases from 200 J to 350 J, under the action of a single constant force, as it accelerates on a smooth level surface, as shown in the figure. What is the work done on the object?




### Solution

Net work done on an object equals the change in kinetic energy of the object. Therefore

$$W_{net} = KE_f - KE_i$$

$$W_{net} = 350\text{ J} - 200\text{ J}$$

$$W_{net} = 150\text{ J}$$

Notice that although neither the force nor the displacement is given in the question, it is possible to find the work done, using the energy given to the object.

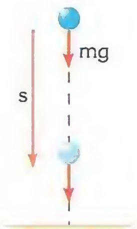



**Figure 6.8** Work done by gravity.

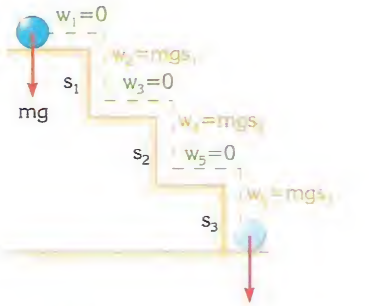



**Figure 6.9** Work done along various path sections which consist of vertical and horizontal sections.

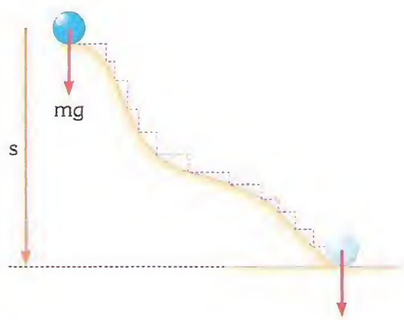



**Figure 6.10** Work done along a series of very small horizontal and vertical sections.

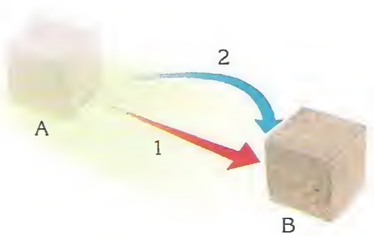



**Figure 6.11** Work done along two different paths on a rough surface.

# 6.3 CONSERVATIVE AND NON-CONSERVATIVE FORCES

## a. Conservative Forces

A force is **conservative** if the work done on an object by this force between two points is independent of the path followed by the object. Otherwise stated, a conservative force always does the same work between two given points, no matter what path is followed in going from the initial to the final position. Therefore, the work done by a conservative force depends only on the initial and final positions of the object.

It can be proved that the gravitational force is conservative.

Consider an object falling through a vertical distance s, as in Figure 6.8. Work done by gravity is

$$\text{Work} = \text{Force} \times \text{distance}$$

$$W_{mg} = mgs$$

Now assume that the object falls through the same vertical distance, but along a different path which consists of vertical and horizontal sections, as shown in Figure 6.9.

$$W = W_1 + W_2 + W_3 + W_4 + W_5 + W_6$$

$$W = 0 + mgd_1 + 0 + mgd_2 + 0 + mgd_3$$

$$W = mg(d_1 + d_2 + d_3)$$

$$W_{mg} = mgs$$

Actually any path between two given points can be divided into a series of very small horizontal and vertical sections, as shown in Figure 6.10.

$$W = mg(d_1 + d_2 + \ldots)$$

$$W = mgs$$

Therefore work done by the gravitational force acting on an object between any two given points is given by

$$W_{mg} = mgd$$

where d is the vertical separation of the points.

Since work done, between two fixed points, by the gravitational force does not depend upon the path taken, gravitational force is a conservative force.

Elastic forces, and electric forces are also conservative forces.

## b. Non-conservative Forces

Contrary to conservative forces, work done by a **non-conservative** force between two points depends on the path connecting the points. Consider an object on the rough surface shown in Figure 6.11. The object is dragged from point A to point B along two different paths: path 1 and path 2. Work done by friction along path

2 is greater than work done by friction along path 1, although the initial and final positions are the same. Therefore work done by friction is path-dependent. Friction is a non-conservative force.

## 6.4 GRAVITATIONAL POTENTIAL ENERGY

Potential energy is stored energy. The **potential energy** of a body is the energy that the body has due to its position or condition. Potential energy is caused by the interaction of two or more bodies via conservative forces.

There is more than one type of potential energy. An object raised above the ground, a stretched spring, and two separated electric charges all have potential energy. The potential energy of an object near the Earth's surface will now be discussed.

In the case of an object raised above the ground, the Earth and the object interact via gravitational force, which is a conservative force. In this case the object is said to have gravitational potential energy ($PE_{gravitational}$).

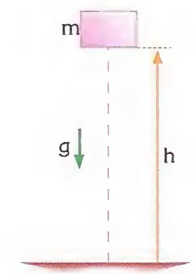



Figure 6.12 A body falling under the influence of gravity

To calculate the gravitational potential energy of an object, a reference point is needed. Consider an object of mass m at a height, h, above the ground as in Figure 6.12. If the ground level is taken to be the reference point, the gravitational potential energy (PE) of the object is

$$PE_{gravitational} = mgh$$

The SI unit for energy is the joule. The potential energy is a scalar quantity. It can have negative values, but the negative value does not indicate a direction. A negative gravitational PE only indicates that the object is below the chosen reference level.

### The Relationship Between Work Done by Gravitational Force and Change in Potential Energy

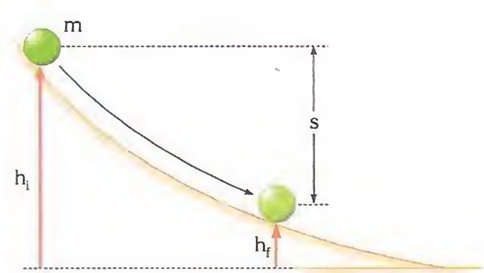



Figure 6.13 An object sliding downwards.

Assume that an object is falling under the influence of gravity, as shown in Figure 6.13. Work done on the object by the gravitational force as it falls a vertical distance of s is

$$W_{mg} = mgs$$

From Figure 6.13, the vertical displacement s is given by $d = h_i - h_f$. Therefore

$$W_{mg} = mgd$$

$$W_{mg} = mg(h_i - h_f)$$

$$W_{mg} = mgh_i - mgh_f = PE_i - PE_f$$

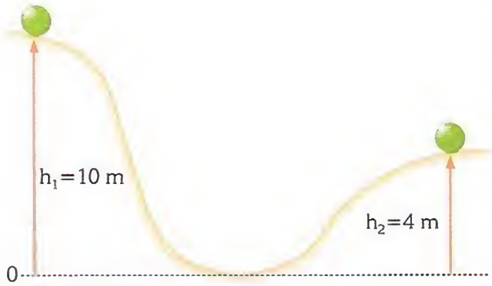
$$W_{mg} = -(PE_f - PE_i)$$

$$W_{mg} = -\Delta PE$$

In words

- When the gravitational force does positive work, $PE_{gravitational}$ decreases.
- Work done by the gravitational force equals the decrease in PE.

A potential energy can be assigned to any conservative force. The conclusions



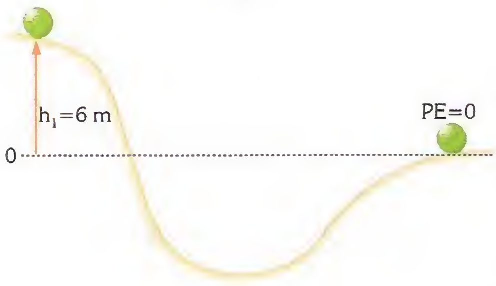




**Figure 6.14** Choice of different reference points. a) falling from 10 m to 4 m. b) falling from 6 m to a level of zero.

above are valid for all conservative forces. For example, when electric forces do positive work, electric potential energy decreases, when elastic forces do positive work spring potential energy decreases.

## Choice of Reference Point is Arbitrary

The change in potential energy does not depend on the chosen reference point. Consider the object in Figure 6.14.a and b. In both cases the object follows the same trajectory, but the chosen reference levels for potential energy are different.

In Figure 6.14.a the object falls from 10 m to 4 m, therefore its initial and final PE values are

$$PE_i = mgh_1 = (3\ kg)(10\ N/kg)(10\ m) = 300\ J$$

$$PE_f = mgh_2 = (3\ kg)(10\ N/kg)(4\ m) = 120\ J$$

the change in PE is –180 J.

In Figure 6.14.b the object falls from 6m to zero level, therefore its initial and final PE values are

$$PE_i = mgh_1 = (3\ kg)(10\ N/kg)(6\ m) = 180\ J$$

$$PE_f = 0$$

the change in PE is –180 J once again.

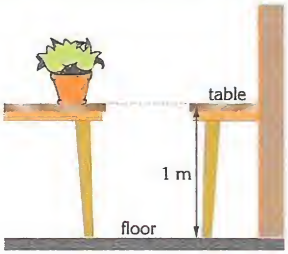
Since the vertical displacement of the object is not affected by the choice of reference point, the gravitational force does the same work in both cases. Therefore the change in PE is the same in both cases. As a result the reference point for gravitational PE can be chosen to be anywhere that is convenient.

## Example 6.3 — Gravitational potential energy

Calculate the potential energy of a plant pot with a mass of 10 kg standing on a 1 m high table, with respect to

a) the table,

b) the floor (Take g=10 N/kg)




### Solution

a) The surface of the table is at the same height as the plant pot. Thus, h = 0 and the potential energy of the pot with respect to this reference level is

$$PE_g = mgh = mg \cdot 0 \quad \text{thus} \quad PE_g = 0$$

b) The pot is 1 metre above the ground, so h = 1 m. Thus, the potential energy of the pot with respect to the floor is

$$PE_g = mgh = (10\ kg)(10\ N/kg)(1\ m)$$

$$\text{thus} \quad PE_g = 100\ J$$

Whenever the pot descends to the level of the floor for any reason, 100 J of potential energy will be transferred to kinetic energy.

## 6.5 CONSERVATION OF MECHANICAL ENERGY

The sum of the potential and kinetic energies of a system is called the total mechanical energy of the system and is denoted as E.

$$E = KE + PE$$

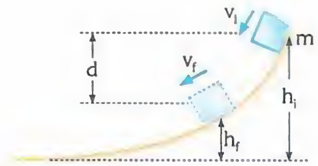



**Figure 6.15** An object sliding down a frictionless surfaces

Figure 6.15 shows an object sliding down a frictionless surface. At the point $h_i$ it has an initial velocity $v_i$ and at the point $h_f$ it has a final velocity $v_f$. Thus, the net work done is the change in the kinetic energy

$$W_{net} = \Delta KE$$

If only gravity does work, $W_{net} = W_{mg}$. Thus,

$$W_{mg} = \Delta KE$$

Since $W_{mg} = -\Delta PE$,

$$-\Delta PE = \Delta KE$$

in the absence of resistive or other non-conservative forces.

In words

- If an object falls under the influence of gravity only, its PE energy decreases and its KE increases. A decrease in PE equals an increase in KE.

Moreover,

$$-\Delta PE = \Delta KE$$

$$-(PE_f - PE_i) = (KE_f - KE_i)$$

$$PE_i + PE_f = KE_f + KE_f$$

$$E_i = E_f$$

This shows that the total mechanical energy of the block at the top and at the bottom is equal.

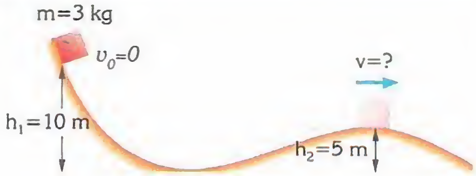
In general the mechanical energy in the absence of friction and other non-conservative forces is conserved.This is called the law of conservation of mechanical energy and is stated generally as

$$\Sigma E_{initial} = \Sigma E_{final}$$

## Example 6.4

Calculate the speed v of a 3-kg object, which is released from a height $h_1=10$ m, at the moment it passes a height $h_2=5$ m. (Neglect any friction effects.)

(Take $g=10$ N/kg)




**Solution**

Let us choose the ground level as the reference point. At the beginning, the object has potential energy. At the second position the object has both potential and kinetic energy. Using the law of conservation of mechanical energy;

$$\sum E_i = \sum E_f$$

$$mgh_1 = mgh_2 + \frac{1}{2}mv^2$$

$$(3\text{ kg})(10\text{ N/kg})(10\text{ m}) = (3\text{ kg})(10\text{ N/kg})(5\text{ m}) + \frac{1}{2}(3\text{ kg})v^2$$

$$v = 10\text{ m/s}$$

## 6.6 HOOKE`S LAW

An elastic substance is capable of returning to its original shape after being stretched or compressed within a limit. When an external force is applied to an elastic object, such as a spring, the object is deformed. When the external force is removed, the object assumes its original size. If the deformation is too great, beyond what is called the **elastic limit** for the substance, the deformation is permanent. The substance loses its elasticity.

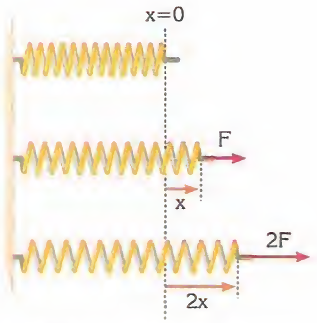



**Figure 6.16** The amount of expansion is proportional to the applied force

A spring has elastic properties. Experimentally it has been found that the elongation of an ideal spring is proportional to the force producing the strain. Figure 6.16 demonstrates this fact. For a given spring,

$$\frac{F}{x} = \frac{2F}{2x} = \frac{3F}{3x} = \text{const}$$

where x refers to the change in the length of the spring under the action of the applied force F.

The constant in the equation is called the "spring constant" of a spring. The spring constant is generally represented by the letter k.

$$\frac{F_{external}}{x} = k$$

For a given spring, the spring constant depends on the size, geometry and type of material.

The proportionality can be written in the form

$$F_{external} = kx$$

The expression written for a spring is applicable to most solids. In words, (within the elastic limit), the force applied to a solid is proportional to the extension produced. This principle is known as **Hooke's law**. The force extension graph for a solid substance is shown in Figure 6.17.

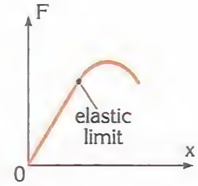



**Figure 6.17** A spring can be compressed or stretched within its elastic limit.

When a spring is stretched or compressed by an external force, a restoring force by the spring, $F_{spring}$, tends to return the spring to its original equilibrium length. The external force and the elastic force are always equal in magnitude but opposite in direction, as shown in Figure 6.18. Therefore

$$F_{spring} = -kx$$

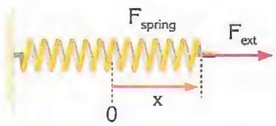



**Figure 6.18** A spring is pulled by a force F.

## Example 6.5 — The restoring force of a spring

A dynamometer of force constant 1000 N/m is used to measure the weight of

a) a mass of 10 kg

b) a mass of 100 kg.

Calculate the displacement (amount of stretching) of the spring. (Take g=10 N/kg)

### Solution

a) When a load of 10 kg is hung from the dynamometer using the equation $W_1 = mg$, the force applied to its spring will be

$$W_1 = (10\text{ kg})(10\text{ N/kg}) = 100\text{ N}$$

Hence, from the Hooke's law equation $F = kx$, the displacement will be

$$100\text{ N} = (1000\text{ N/m})x \quad \text{thus} \quad x = 0.1\text{m}$$

b) Here, $W_2 = (100\text{ kg})(10\text{ N/kg}) = 1000\text{ N}$.

Similarly, from Hooke's law

$F = kx$ where $1000\text{ N} = (1000\text{ N/m})x$ thus $x = 1\text{ m}$

This amount of displacement must be above the elastic limit and such a load will break the dynamometer.

## 6.7 ELASTIC POTENTIAL ENERGY

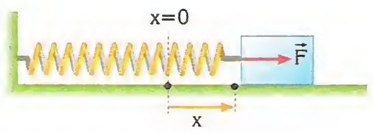



**Figure 6.19** A load is attached to a spring and pulled by a force F.

Forces acting on a spring are conservative. Therefore a potential energy can be assigned to a spring.

Assume that a spring is stretched a distance x by an external force, as shown in Figure 6.19. Work done by the external force in pulling the spring is stored as the **elastic potential energy** of the spring. The work done by the external force equals the area under the force extension graph in Figure 6.20.

$$W_{external} = \text{Area under F-x graph}$$

$$W_{external} = \frac{1}{2}(kx)(x)$$

$$W_{external} = \frac{1}{2}kx^2$$

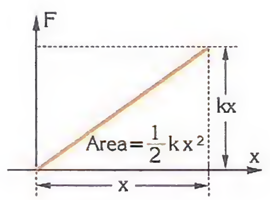



**Figure 6.20** Area under the F-x graph gives the work done.

This amount of work done by the external force is stored in the spring as the elastic potential energy of the spring. Therefore, the elastic potential energy stored in a compressed or stretched spring is given by

$$PE_{spring} = \frac{1}{2}kx^2$$

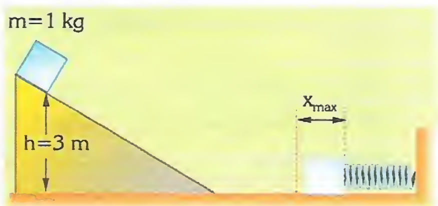
Similar to other conservative forces, positive work done by a spring equals the decrease in the spring's potential energy:

$$W_{spring} = -\Delta PE_{spring}$$

In the absence of non-conservative forces, total mechanical energy of a system does not change.

### Example 6.6 — Conservation of mechanical energy

By how much can a 1-kg object, which is released from a height of 3 m on an inclined surface, compress a spring of force constant k = 240 N/m, shown in the figure? (Neglect any friction effects; take g=10 N/kg)




**Solution**

Since friction is neglected, the mechanical energy in this system is conserved.

The object initially has gravitational potential energy. When the spring is compressed to a maximum, the speed of the object will be zero, hence it will have no kinetic energy. Here, the initial gravitational potential energy of the object will completely be converted to potential energy of the spring because of the conservation of mechanical energy.

Therefore;

$$\Sigma E_{initial} = \Sigma E_{final}$$

$$mgh = \frac{1}{2}kx_{max}^2$$

$$(1\text{ kg})(10\text{ N/kg})(3\text{ m}) = \frac{1}{2}(240\text{ N/m})x_{max}^2$$

$$x_{max} = 0.5\text{ m}$$

## 6.8 NON-CONSERVATIVE FORCES AND THE GENERALISED WORK - ENERGY THEOREM

When kinetic friction does work on an object, the total mechanical energy of the object decreases. Therefore mechanical energy is not conserved in the presence of **resistive forces**.

Actually energy in its widest sense is never lost. The loss in the mechanical energy caused by friction appears as thermal energy (heat). The amount of thermal energy produced by friction equals the amount of lost mechanical energy.

The relationship between work done by friction (or other non-conservative forces) and the change in mechanical energy can be established as follows:

Consider an object sliding down a rough inclined plane, as shown in Figure 6.21. As the object slides down, the gravitational force and the friction force does work on it. A change in the kinetic energy of the object equals the net work done on it.

$$W_{net} = \Delta KE$$

$$W_{mg} + W_{friction} = \Delta KE$$

Since gravity is conservative, ($W_{mg}$) can be replaced by ($-\Delta PE_{gravitational}$)

$$W_{friction} - \Delta PE_{gravitational} = \Delta KE$$

Rearranging gives

$$W_{friction} = \Delta KE + \Delta PE$$

$$W_{friction} = \Delta E$$

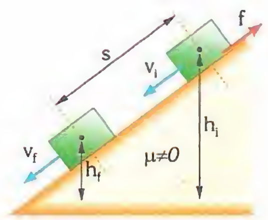



Figure 6.21 An object sliding down a rough surface.

Therefore, in Figure 6.21, work done by friction on the object equals the change in total mechanical energy of the object. Since the work done by friction is negative, the total mechanical energy of the system decreases. The lost mechanical energy is transferred into thermal energy by the force of friction. The amount of thermal energy produced also equals the magnitude of the work done by friction.

In the most general case,

$$W_{nonconservative} = \Delta E$$

$$W_{nonconservative} = E_f - E_i$$

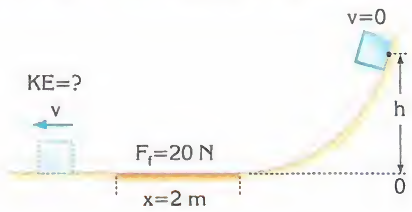
$$E_i + W_{nc} = E_f$$

The net work done by non-conservative forces acting on a physical system equals the change in the mechanical energy of that system. If the non-conservative work done on the system is negative, the total mechanical energy of the system decreases. If the non-conservative work done on the system is positive, the total mechanical energy of the system increases.

In general, work implies energy transfer. Doing work on a physical system means transferring energy to or from the system.

## Example 6.7 — Change in mechanical energy

The object in the figure is calculated to have 100 J of gravitational potential energy while it is at rest above the ground level on a smooth inclined track. When it is released it starts to slide down the track. The track is smooth except for a 2-m horizontal section over which a friction force of 20 N acts on the object.




a) What is the final kinetic energy of the object?

b) What happens to the lost mechanical energy?

### Solution

a) The change in the total mechanical energy equals the work done by friction

$$E_i + W_{nc} = E_f$$

The initial mechanical energy of the object is purely gravitational potential energy. The final total mechanical energy of the object is in the form of kinetic energy only.

b) The only nonconservative force in the problem is the force of friction. Therefore,

$$PE_i + W_{friction} = KE_f$$

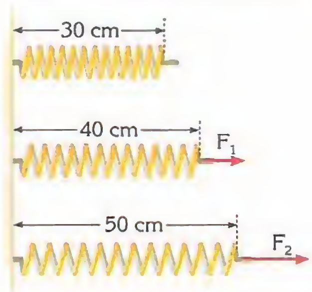
$$100\text{J} - (10\text{N})(2\text{m}) = KE_f$$

$$KE_f = 100\text{J} - 40\text{J}$$

$$KE_f = 60\text{J}$$

## Example 6.8 — Non-conservative forces

What work is needed to stretch a spring's length from 40 cm to 50 cm, if its unstretched length is 30 cm? The force constant of the spring is given as 800 N/m.




### Solution

The initial and final extensions of the spring are

$x_i$ = 40 cm - 30 cm = 10 cm

$x_f$ = 50 cm - 30 cm = 20 cm

Work done in stretching the spring is given by:

$$W_{nc} = \Delta E$$

$$W_{nc} = E_f - E_i$$

$$W_{nc} = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2$$

$$W_{nc} = \frac{1}{2}(800\ \text{N/m})(0.2\ \text{m})^2 - \frac{1}{2}(800\ \text{N/m})(0.1\ \text{m})^2$$

$$W_{nc} = 16\ \text{J} - 4\ \text{J}$$

$$W_{nc} = 12\ \text{J}$$

The elastic potential energy stored in the spring increases from 4J to 16 J during the stretching process. This energy must be supplied by the external force. Therefore the external force stretching the spring must have done 12 J of work.

## Energy Conservation in General

Mechanical energy is one of many different forms of energy. There are many other forms of energy such as thermal, chemical, electromagnetic, and nuclear energies.

In general, energy cannot be created or destroyed. However, it can be converted from one form to another. Friction converts mechanical energy into thermal energy. An electric generator converts mechanical energy into electric energy and thermal energy, a solar cell converts electromagnetic energy from the Sun into chemical energy stored in a battery, etc. During all these processes, the total energy remains constant. A decrease in one form of energy is always accompanied by an equal increase in other forms of energy. Therefore the total energy of an isolated system never changes. This statement is known as the law of **conservation of energy**.

# 6.9 POWER

**Power** in mechanics is defined as the time rate of doing work or "the work done in unit time" and is denoted as p. Hence, if an amount of work W is done in a time interval t, the power which does this rate of work is given as;

$$P = \frac{W}{t}$$

In the SI unit system the unit of power is the Watt (W). 1 watt equals 1 joule per second.

$$1\ \mathrm{W} = 1\ \frac{\mathrm{J}}{\mathrm{s}}$$

Another widely used unit for power is horsepower (hp), it is especially used for heavy machinery.

$$1 \text{ horse power } = 1 \text{ hp} = 746 \text{ W} \approx 0.75 \text{ kW}$$

The power formula can be written as

$$P = \frac{W}{t}$$

$$P = \frac{Fs}{t} = F\left(\frac{s}{t}\right)$$

However, displacement over time is velocity. Therefore

$$P = Fv$$

where the force is parallel to the velocity. The product of force acting on an object and the instantaneous velocity of the object equals the power transferred to the object at a given instant.

## Efficiency

When energy is converted from one form into another, an amount of total energy is invariably wasted as heat (thermal energy). This thermal energy is often called "waste" since it is not possible to use it again to produce useful work. The energy converted to thermal energy usually ends up heating the surroundings. Thus, a system must be given more energy than it ideally requires to do work.

For example, in raising a load above the ground, an electric winch does work against gravity. This work done against gravity is stored as the potential energy of the load. This energy is still useful, since it will be used again. However, a working winch gets hot. It produces thermal energy. Therefore, when the winch actually lifts the load, the total electric energy given to the winch is the work done by the winch against gravity, plus the thermal energy produced by the winch.
**Efficiency** is defined as

$$\text{efficiency}=\frac{\text{useful energy output}}{\text{total energy input}}=\frac{\text{useful power output}}{\text{total power input}}$$

$$e=\frac{E_{useful}}{E_{total}}=\frac{P_{useful}}{P_{total}}$$

for a physical system. Efficiency is a dimensionless ratio. Percent efficiency is

$$\%e = e \times 100$$

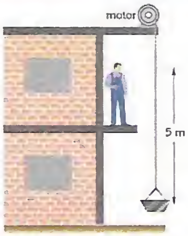
## Example 6.9 — Power and efficiency

An electric motor can lift a load of 50 kg to a height of 5 m, at a constant speed, in 4 s.

a) Calculate the mechanical power of the motor in Watts and in horse power.

b) If 1000 W of electric power is needed to work the motor, find the efficiency of the motor.

(Take g=10 N/kg; 1 horsepower (hp) ≅ 750 W)




**Solution**

a) The work done by the motor will be transferred to potential energy of the load. By ascending 5 m, the 50-kg load gains an energy of;

$$PE = mgh = (50\text{ kg})(10\text{ N/kg})(5\text{ cm}) = 2500\text{ J}$$

Therefore, the motor transfers 2500 J of energy in 4 s. Hence, the power of the motor is found from

$$P=\frac{PE}{t}=\frac{2500\text{ J}}{4\text{ s}} \quad \text{thus} \quad P = 625\text{ W}$$

Since 625 Watt=0.625 kW and 1 hp=0.75 kW
The amount of energy can be found in horse power as

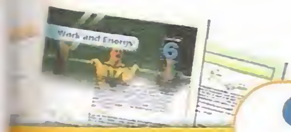
$$P=\frac{0.625}{0.75} \quad \text{thus} \quad P = 0.83\text{ hp}$$

b) The efficiency of a system is equal to the ratio of useful energy output to total energy input. In the question the electric power needed for the motor to work is 1000 W. Whereas, it can be seen that the motor has only 625 W of mechanical power when lifting the load. Considering the power to be the energy per unit time, the efficiency of the motor is found using

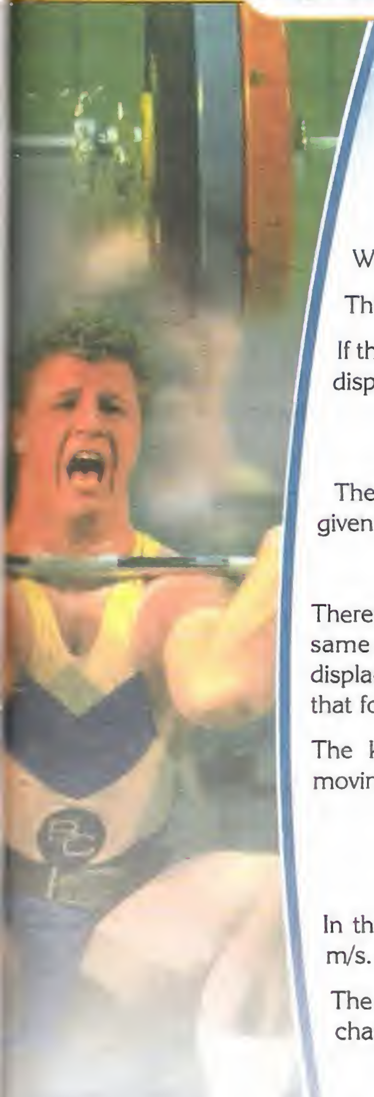
$$\text{Efficiency}=\frac{\text{Useful energy output}}{\text{Total energy input}}=\frac{\text{Useful power output}}{\text{Total power input}}$$

$$\text{Efficiency}=\frac{625\text{ W}}{1000\text{ W}}$$

$$\text{Efficiency} = 0.625 \quad \text{or} \quad 62.5\%$$




# Summary



Work is defined as the product of a force and the distance through which it moves a body in the direction of that force.

$W = Fs$ (in which force and displacement are in parallel directions)

Work is a scalar quantity.

The SI unit for work is the Joule (J).

If the angle between the force vector F and the displacement vector, s, is θ, the work done is

$$W = Fs \cos\theta$$

The area under the force-position graph is given by

$$\text{Area} = \text{Force x distance}$$

Therefore, if force and displacement lie along the same axis, the area under the force-displacement graph equals the work done by that force.

The kinetic energy of an object of mass m moving at a velocity v is given by

$$KE = \frac{1}{2}mv^2$$

In the formula mass is in kg and velocity is in m/s.

The net work done on an object equals the change in kinetic energy of the object.

$$W_{net} = \Delta KE$$

This expression is known as the work-kinetic energy theorem.

A force is conservative, if the work done on an object by this force between two points is independent of the path followed by the object.

Contrary to conservative forces, work done by a non-conservative force between two points depends on the path connecting the points.

The gravitational potential energy of an object as it falls a vertical distance of s is

$$W_{mg} = mgs$$

Work done by a gravitational force equals the decrease in PE. That is;

$$W_{mg} = -\Delta PE$$

The mechanical energy in the absence of friction and other non-conservative forces is conserved.This is called the law of conservation of mechanical energy and is stated generally as;

$$\sum E_{initial} = \sum E_{final}$$

The force applied to a solid is proportional to the extension produced. This principle is known as Hooke's law. That is

$$F_{external} = kx$$

The elastic potential energy stored in a compressed or stretched spring is given by

$$PE_{spring} = \frac{1}{2}kx^2$$

The net work done by non-conservative forces acting on a physical system equals the change in the mechanical energy of that system.

$$E_i + W_{nc} = E_f$$

In general, energy cannot be created or destroyed. However, it can be converted from one form to another.

Power in mechanics is defined as the time rate of doing work or "the work done in unit time" and is denoted as p.

$$P = \frac{W}{t}$$

Efficiency is defined as

$$\text{efficiency} = \frac{E_{useful}}{E_{total}} = \frac{P_{useful}}{P_{total}}$$



# QUESTIONS AND PROBLEMS

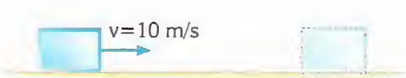
6.1 Work

1. Explain the difference between the meaning of the word "work" in physics and in daily life.

2. Does every force do work? Answer this question by considering the forces acting on yourself at the moment.

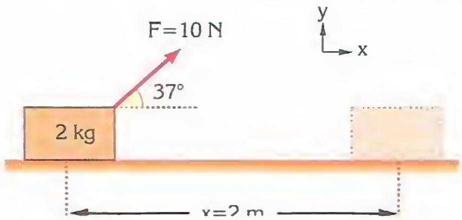
3. Give two examples of forces which act on an object but do not do any work.

4.



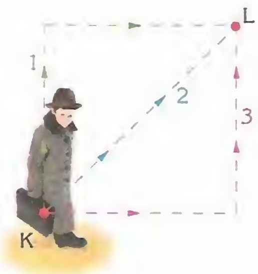

How much work is done if a force of 10 N moves an object a distance of 10 m?

5.




A 2-kg object which rests on a frictionless horizontal plane, is acted on by a force of 10 N at an angle of 37° with the horizontal, as shown in the Figure. If the object moves a distance of 2 m on the horizontal plane, what is the work done by force $\vec{F}$?

6. A boy takes his 5 kg bag to point L which is 3 m vertically upwards along 3 different paths, as shown in the figure. What is the relationship between the work done while lifting the bag along the 3 different paths? (Take g=10 N/kg)

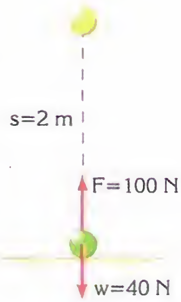



7. If a force of 100 N acts on an object weighing 40 N and lifts it to a height of 2 m, as shown in the figure.

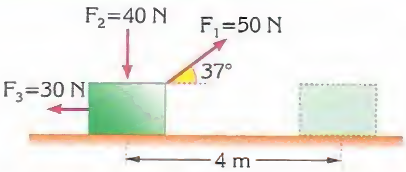
a) What is the work done by the force of 100 N?

b) What is the work done by the weight of the object?

c) What is the net work done?



**8.**

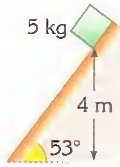



An object moves horizontally 4 m under the effect of forces $F_1$ = 50 N, $F_2$ = 40, and $F_3$ = 30 N, as shown in the figure. Find

a) the work done by each force

b) the net work done by the net force.

**9.** A 5 kg object is released from a height of 4 m on a smooth inclined plane, as shown in the figure. When it reaches the bottom of the inclined plane, find

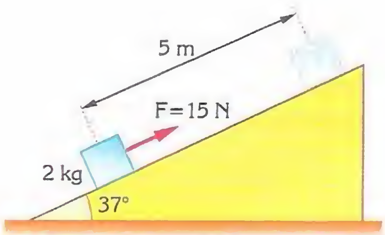



a) the work done by the normal force

b) the work done by the gravitational force

(Take g = 10 N/kg; sin 53°=0.8; cos 53°=0.6)

**10.**




A force F = 15 N is applied to a 2 kg block resting on an inclined frictionless plane. If the block moves 5 m under the effect of this force, find;

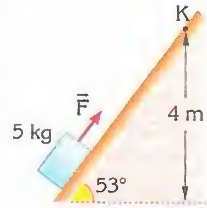
a) the work done by force $\vec{F}$

b) the work done by the gravitational force.

c) Does the reaction force from the surface do any work?

(Take sin 37°=0.6; cos 37°=0.8)

**11.** By applying a force $\vec{F}$, as shown in the figure, a 5 kg object is moved at a constant speed from the bottom of an inclined plane to point K on the smooth inclined plane. Find



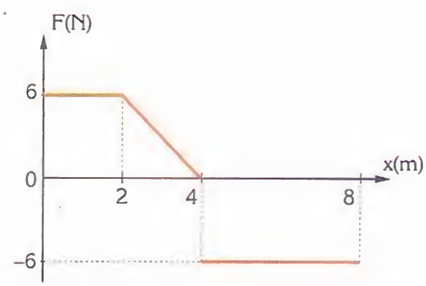

a) the work done by force $\vec{F}$

b) the work done by the normal force

c) the work done by gravity

(Take sin 53°=0.8; cos 53°=0.6)

**12.**




The force versus position graph of an object on a horizontal frictionless plane is shown in the figure. If the force is applied to the object parallel to the direction of its motion, find the work done by the force

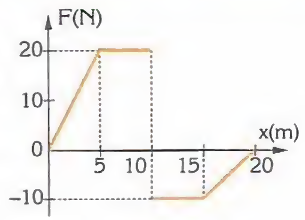
a) between x = 0 and x = 2 m,

b) between x = 2 m and x = 4 m,

c) between x = 4 m and x = 8 m,

d) between x = 0 and x = 8 m

13. The force versus position graph of an object on a horizontal frictionless plane is as shown in the figure. If the force is applied to the object parallel to the direction of its motion, find the net work done between x=0 and x=20 m




14. What are the following amounts of energy in joules?

a) 1 kJ = .............................. J

b) 10 MJ = .............................. J

6.2 Kinetic Energy and the Work-KE Theorem

15. What does kinetic energy depend on?

What is the SI unit of energy?

16. Which one has more kinetic energy; a train carriage or a car, if both are moving at the same speed?

17.


How many joules is the kinetic energy of a 1 kg toy aeroplane;

a) when its speed is 4 m/s?

b) when its speed is 10 m/s?

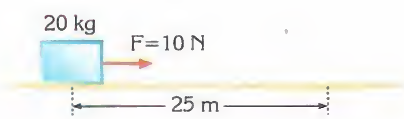
18.


One of two lorries is half the weight of the other, but is moving at twice the speed of the other. Which of the lorries has more kinetic energy?

19. If the speed of a 20 g bullet which is shot out of a gun is 200 m/s, what is the kinetic energy of the bullet?

20. The kinetic energy of a 0.5 g rain drop which is falling at its terminal speed is $10^{-3}$ J. What is the terminal speed of the rain drop?

21.




While initially resting on a frictionless horizontal plane, a 20 kg case is pulled 25 m by a 10 N force.

a) What is the work done?

b) What is the speed of the case at the end of 25 m?

c) How many seconds does it take for the case to reach this speed?

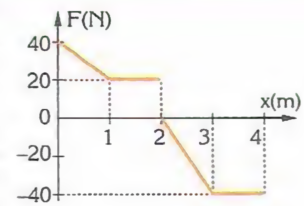
22. At the instant a 6 kg object starts to move from $x = -1$ m towards the +x direction at a velocity of 4 m/s, a force of 24 N is applied to the object until $x = 5$ m. If there is no friction between the object and the floor

a) what is the change in the kinetic energy of the object?

b) what is the velocity of the object at $x = 5$ m?

c) If the initial velocity of the object was zero, what would its velocity be at $x = 5$ m?

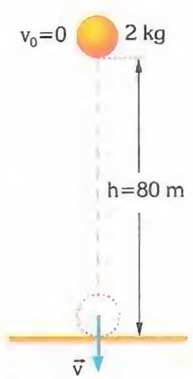
23. The force–position graph of a 1 kg object, experiencing a horizontal force $\vec{F}$ on a smooth horizontal plane, is shown in the figure. If the object reaches a velocity of 4 m/s after it covers 4 m, find




a) the net work done between 0 – 4 m

b) the initial velocity of the object.

24. A 2 kg object is released from a height of 80 m.




a) What are the forces acting on the object?

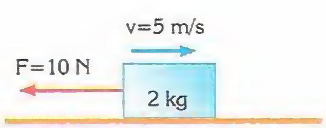
b) Find the work done by each force.

c) What is the change in the kinetic energy of the object during its flight?

d) What is the velocity of the object at the moment it strikes the ground?

(Take g=10 N/kg and neglect any friction effects.)

25.




A 2 kg block is moving on a horizontal frictionless surface at a velocity of v=5 m/s. A force of F=10 N is applied to it in the opposite direction to its motion, as shown in the figure. How many joules of work has been done by the force when the block stops?

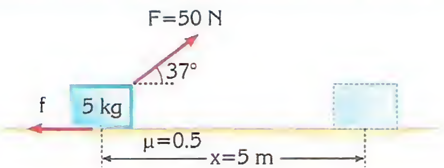
**26.** A 2 kg object is moving on a smooth horizontal surface at a velocity of 10 m/s. A horizontal force F is applied to it in the same direction as its motion in 4 s. If the object gains an acceleration of 0,5 m/s$^2$, find

a) the work done by the force $\vec{F}$

b) the change in the kinetic energy of the object.

## 6.3 Conservative and Non-conservative Forces

**27.**




A force of 50 N is applied to a 5 kg block resting on a horizontal plane, as shown in the figure. The coefficient of kinetic friction between the object and the surface is $\mu = 0.5$. If the block moves 5 m under the effect of this force, calculate

a) the work done by the friction force f

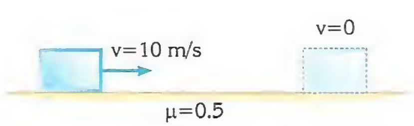
b) the work done by force $\vec{F}$

c) the net work done by the net force

d) Its velocity after it travels 5 m

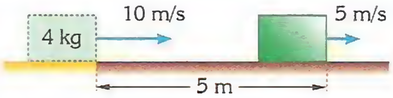
(Take cos37°=0.8; sin37°=0.6 ; g=10 N/kg)

**28.**




An object is pushed with a velocity of 10 m/s onto a horizontal surface. If the coefficient of kinetic friction between the object and the surface is 0.5. How far does it move before it stops?

**29.**

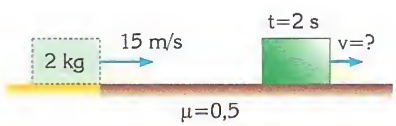



A 4 kg object moving at a speed of 10 m/s on a smooth horizontal plane enters a rough section of the plane. After it covers a distance of 5 m, its velocity decreases to 5 m/s. Find

a) the work done by the friction force

b) the coefficient of kinetic friction(Take g=10 m/s$^2$)

**30.**

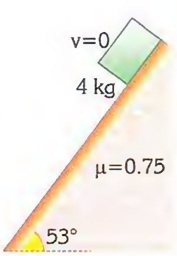



A 2 kg object moving at a velocity of 15 m/s enters a rough surface, as shown in the figure. If the coefficient of kinetic friction between the object and the surface is 0.5, find

a) the work done by the friction force in 2 s

b) the velocity of the object after 2 s (Take g=10 m/s$^2$)

**31.** A 4 kg object is released from a rough inclined plane, as shown in the figure.The coefficient of kinetic friction between the object and the inclined plane is $\mu = 0.75$. If the object moves 2 m, find

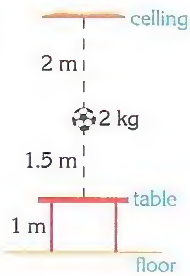



a) the work done by the friction force

b) the change in the kinetic energy of the object

## 6.4 Gravitational Potential Energy

**32.** What is the potential energy of the ball positioned in the room shown in the figure



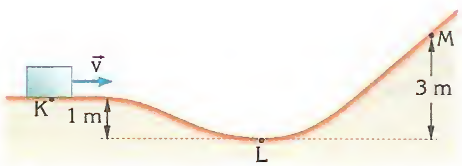

a) With reference to the ceiling?

b) With reference to the table?

c) With reference to the floor?

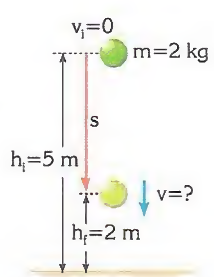
(Take $g = 10\ m/s^2$)

**33.**




A 20 kg object is moving on a smooth path, as shown in the figure. Find its gravitational potential energies at points L and M with reference to point K (Take $g=10\ m/s^2$)

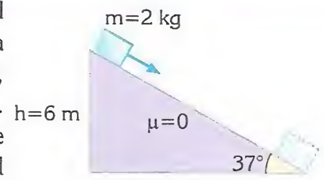
**34.** A 2 kg object is released from a height of 5 m. When it falls to a point 2 m above the ground, as shown in the figure,




a) how much work is done by gravity ?

b) What is the change in the potential energy?

**35.** A 2 kg object is released from the top of a smooth inclined plane, which is 6 m high. When it reaches the bottom of the inclined plane

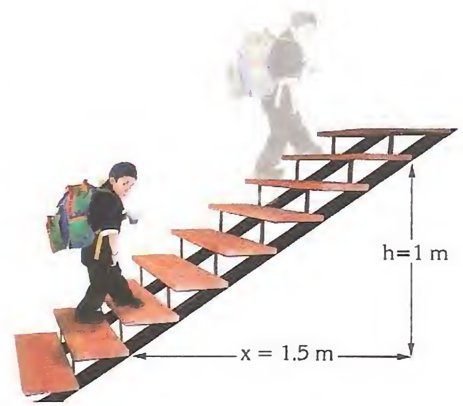



a) what is the work done by the gravitational force ?

b) what is the change in its potential energy?

(Take $\sin 37° = 0.6$ ; $\cos 37° = 0.8$)

**36.**




A 60 kg boy is climbing 1 m to the top of the stairs with a constant speed as shown in the figure.

a) How many Joules of potential energy does the boy gain?

b) How many Joules of work does the boy do against gravity?

**37.**


What work is done in bringing the uniform object from the horizontal position to the vertical position, as shown in the figure?

**38.**

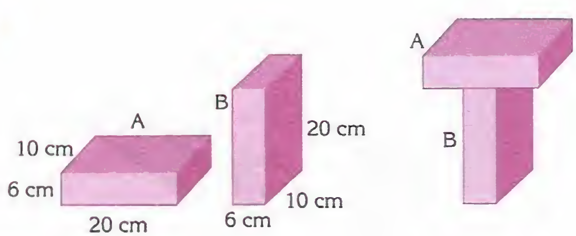


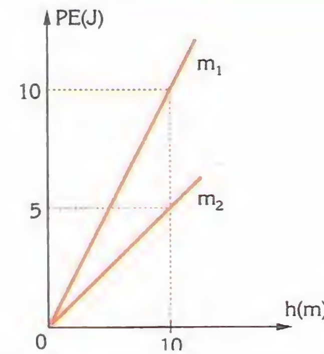

Figure-I Figure-II

Two homogeneous objects A and B of dimensions 20 cm, 10 cm and 6 cm rest in two different configurations, as shown in Figure-I. Later, object A is placed on top of object B, as shown in Figure-II. If the mass of each object is 10 kg

a) what are the gravitational potential energies of the objects in Figure-I relative to the floor?

b) what work is done to obtain the configuration in Figure-II?

**39.**




The graphs of potential energy versus height from a chosen reference level, of masses $m_1$ and $m_2$ are shown in the figure.

a) How many kg are the masses $m_1$ and $m_2$?

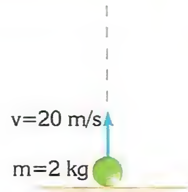
b) What are the potential energies of these masses when they are lifted to a height h = 20 m?

### 6.5 Conservation of Mechanical Energy

**40.** An object is released 5 m above the ground. What is its velocity at the moment it strikes the ground? (Take $g=10\ m/s^2$)

**41.** A ball with a mass of 2 kg is dropped from the top of a building and strikes the ground at a velocity of 10 m/s. What is the height of the building from which the ball is dropped?

**42.** A 2-kg ball is thrown upwards at a velocity of 20 m/s.

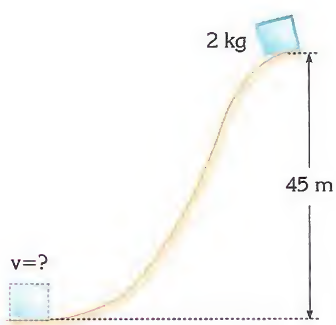



a) What is the potential energy when it is at its maximum height?

b) What is the velocity of the ball when it is 15 m above the ground? (Take $g=10\ m/s^2$)

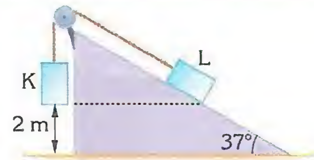
**43.** When a 4-kg ball is thrown upwards at 40 m/s, at what height is the potential energy equal to the kinetic energy.

**44.**



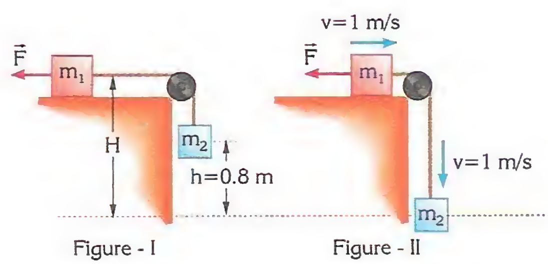

A 2-kg object slides down a smooth hill, as shown in the figure. What is the velocity of the object at the bottom? (Take $g=10m/s^2$)

**45.** The frictionless system shown in the figure, composed of two identical objects K and L, is released from rest.



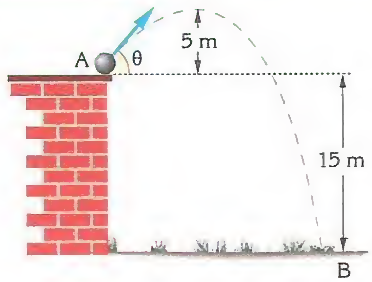

What is the velocity of object K, in m/s, when it strikes the ground? (Take $\sin 37°=0.6$; $g=10$ N/kg)

**46.**




The system in the figure, consisting of two objects of masses 5 kg and 3 kg, are attached to each other with the aid of a pulley. The system arrives at the position in figure II from the position in figure I, under the effect of force $\vec{F}$. If the speed of each object in figure II is 1 m/s, how many newtons is force $\vec{F}$?

47.




An object of mass 400 g is launched, as shown in the figure, making a certain angle with the horizontal. If the maximum height the object can reach is 5 m and its kinetic energy at its maximum height is 20 J, what is

a) the vertical component of the initial velocity

b) the kinetic energy of the object at the moment of launch

c) the work done by gravity from the point where the object is launched to the point where it lands.

d) the kinetic energy of the object at the moment it strikes the ground?

6.6 Hooke's Law

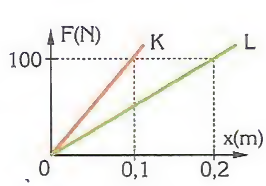
6.7 Elastic Potential Energy

48. If a spring has a spring constant of 800 N/m, find its potential energy when it is compressed by

a) 10 cm.

b) 20 cm.

49. In the figure the force-position graphs for two different springs are shown. Calculate the potential energy stored in each spring when they are stretched x = 0.4 m




50.

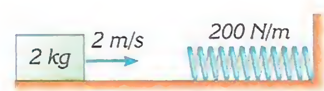

An 80 kg man steps on a spring, as shown in the figure. If he compresses the spring by 0.8 cm, calculate

a) the spring constant of the spring

b) the work done by the weight of the man

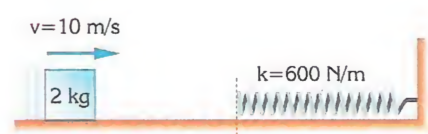
c) the energy stored in the spring.

51.




An object of mass 2 kg moving with a velocity of 2 m/s compresses a spring with a spring constant of 200 N/m, as shown in the figure. Calculate the maximum amount of compression of the spring.

52.

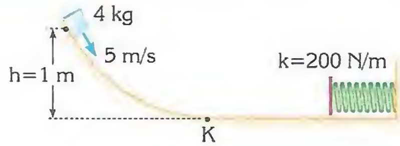



A 2 kg object is launched at a velocity of v=10 m/s towards a spring with a spring constant of k=600 N/m, as shown in the figure. Calculate

a) the compression of the spring at the moment the velocity of the object is 5 m/s,

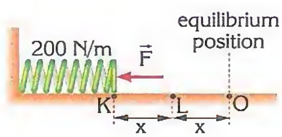
b) the maximum amount of compression of the spring. (Neglect any friction effects.)

**53.**




A 4 kg object at a height of 1 m is pushed with a velocity of 5 m/s towards a spring, as shown in the figure. If the surface is frictionless and the spring constant is 200N/m, what is the maximum compression of the spring?

**54.** When a spring is compressed by a force F from point L to point K, as shown in the figure, its potential energy increases by 12 J . If the spring constant is 200 N/m find the distance, x.

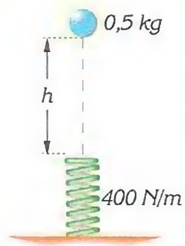



**55.** An archer places an arrow in a bow and pulls the bowstring 60 cm back from its equilibrium position, by applying a force of 300 N. By considering the bowstring to be a spring, find

a) the spring constant,

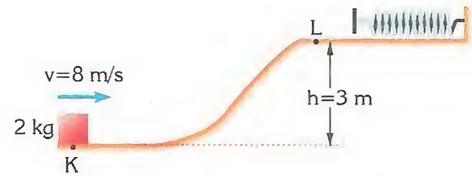
b) the potential energy stored in the bowstring when stretched.

**56.**




A 0.5 kg mass released from a height h compresses a spring by 0.1 m, as shown in the figure. If the spring constant is 400 N/m, find the height, h. (Take $g=10\ m/s^2$)

**57.**

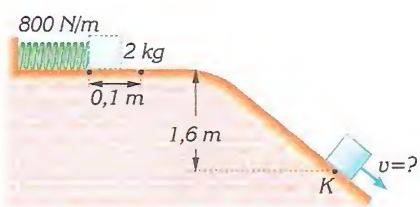



A 2 kg object which is launched at a velocity of 8 m/s from point K, passes point L, which is at a height of 3 m, at velocity $v_L$ and compresses the spring.

a) What is the velocity of the object at point L?

b) If the spring constant is 200 N/m, what is the maximum amount of compression of the spring? (Neglect any friction effects.)

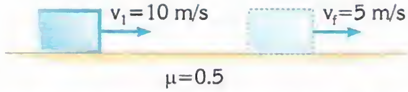
**58.**




A spring which has a spring constant of 800 N/m is compressed 0.1 m by a 2 kg object, as shown in the figure. If the whole path is smooth, find the velocity, v of the object at point K? (Take $g = 10\ m/s^2$)

### 6.8 Non-conservative Forces and the Generalised Work-Energy Theorem

**59.**

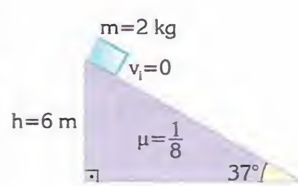



A 1 kg object is pushed at a velocity of 10 m/s on a rough surface, as shown in the figure. The coefficient of kinetic friction between the object and the surface is 0.5.

a) How far does the object go before its velocity decreases to 5 m/s

b) What is the maximum distance that the object can cover?

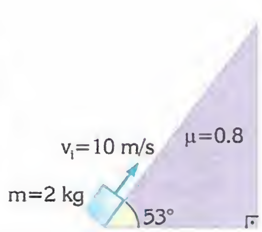
**60.** A 2 kg object is released from a height of 6 m on an inclined plane, as shown in the figure. If the coefficient of kinetic friction everywhere between the surface and the object is $\frac{1}{8}$



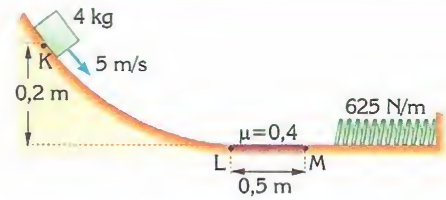

a) what is its velocity when it reaches the bottom of the inclined plane?

b) how many metres does it cover in the horizontal direction?

**61.** A 2 kg object is pushed upwards at a velocity of 10 m/s from the bottom of a rough inclined plane, as shown in the figure. At what height does it stop if $\mu=0.8$?




**62.**



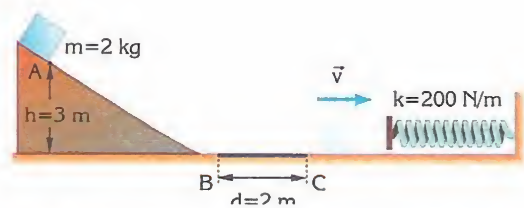

A 4 kg object is pushed with a velocity of 5 m/s from a height of 0.2 m towards a spring with a spring constant of 625 N/m, as shown in the figure. If it covers a distance of 0.5 m on a rough horizontal surface between points L and M, where $\mu = 0.4$, find

a) the velocity of the object at point M

b) the maximum compression of the spring

(Take $g = 10$ m/s$^2$)

**63.**




A 2 kg object released from point A on the inclined plane shown in the figure, passes through the rough path section BC. At point C it has a velocity v, with which it continues along the rest of the smooth path before compressing the spring. If the coefficient of kinetic friction between points B and C is 0.6

a) find the velocity v of the object at point C.

b) If the spring constant is 200 N/m, find the maximum compression.

### 6.9 Power

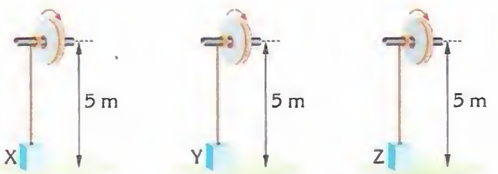
**64.** In a building under construction, a motor is used to lift a 40 kg load to a height of 8 m, at a constant velocity, in 20 seconds. find

a) the work done by the motor

b) the power of the motor

**65.** An electric motor is used to lift a 50 kg load. If the load is lifted 3 m upwards in 5 seconds at a constant speed, find the power of the motor.

**66.**




Three electric motors are used to lift 3 objects each of mass 6 kg to a height of 5 m, as shown in the figure. Object X is lifted to this height in 3 s. Object Y is lifted in 1 s and object Z is lifted in 2 s. Find the powers $P_x$, $P_y$ and $P_z$ of the motors.

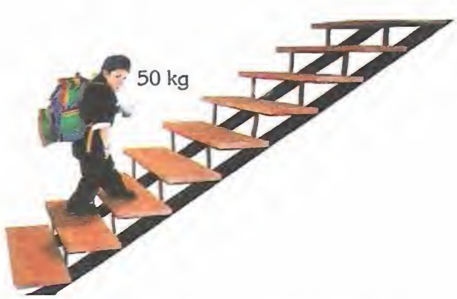
**67.** A speedboat of mass 750 kg increases its velocity from rest to 100 m/s in 10 s. Neglecting the friction forces, find

a) the work done by the speedboat in 10 s

b) the power of the speedboat in horsepower (hp)

(1 hp ≅ 750 Watt)

**68.**




What is the power of a child, weighing 50 kg, who climbs 50 steps, each being 20 cm high, in 10 s?

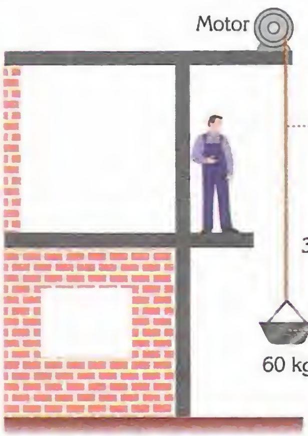
**69.** A lorry whose mass is 8 tons can reach a velocity of 36 km/h from 18 km/h in 10 s.

a) What is the increase in the kinetic energy of the lorry?

b) How many horse power is the engine of the lorry?

(1 hp ≅ 750 Watt)

**70.**




If a motor can lift a 60 kg load to a height of 3 m in 10 s by consuming 2000 J of energy, what is the power and the efficiency of this motor?

**71.** An 800 kg lift ascends 4 m in 1 s. If the efficiency of the motor pulling the lift is 64%, how many watts of power does the motor dissipate?

CHAPTER 7

# Momentum and Impulse

*You wouldn't want to get in the way of a heavy moving truck or a speeding bullet. Both are hard to stop, but for different reasons . The truck is moving slowly compared to the bullet but it has a great mass. The bullet is light but it has a very high velocity. In this chapter momentum, impulse and collisions will be discussed. The motion of rockets in space will also be discussed.*



**Figure 7.1** An object of mass m moving at a velocity v has a momentum of $\vec{p} = m\vec{v}$

## 7.1 MOMENTUM

The **momentum** of an object is defined as the product of its mass and velocity. In other words, if an object having a mass m is moving at a velocity $\vec{v}$, as shown in Figure 7.1, it is said to have a momentum of $m\vec{v}$.

$$\text{momentum} = \text{mass x velocity}$$

$$\vec{p} = m\vec{v}$$

Momentum is a vector quantity, that is, it has both magnitude and direction and is denoted as p. It acts in the same direction as the velocity. The SI unit of momentum is kg·m/s.

## Momentum and Its Relationship to Force

When a force, $\vec{F}$ is applied to an object moving with a velocity $\vec{v}_i$ in a time interval $\Delta t$, as shown in Figure 7.2, the object accelerates and reaches a velocity $\vec{v}_f$. Therefore, it undergoes a change in its momentum. The relationship between the change in the momentum of such an object and the force applied to it was expressed by Newton as


**Figure 7.2** Applied force F causes a change in momentum.

$$\vec{F} = \frac{\text{change in momentum}}{\text{time interval}} = \frac{\Delta\vec{p}}{\Delta t}$$

Newton first stated the second law of motion in this mathematical form, not as F = ma, he obtained the expression for the second law of motion as follows.

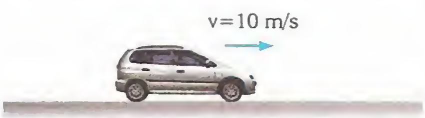
$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t} = \frac{\vec{p}_f - \vec{p}_i}{\Delta t} = \frac{m\vec{v}_f - m\vec{v}_i}{\Delta t}$$

$$\vec{F} = \frac{m(\vec{v}_f - m\vec{v}_i)}{\Delta t} \quad \text{where } \vec{a} = \frac{\vec{v}_f - \vec{v}_i}{\Delta t}$$

$$\vec{F} = m\vec{a}$$

## Example 7.1 Momentum and change in momentum

A car having a mass of 900 kg is moving due east at a velocity of 10 m/s. Calculate




a) the momentum of the car
b) the change in momentum of the car when its velocity increases to 30 m/s,

**Solution**

a) From the equation of momentum

$$p = mv = (900 \text{ kg})(10 \text{ m/s})$$

thus $p = 9000$ kgm/s

Since the momentum has the same direction as the velocity, the momentum of the car is also due east.

b) The initial and final moment a of the car are

$$p_i = mv_i = (900 \text{ kg})(10 \text{ m/s}) = 900 \text{ kgm/s}$$

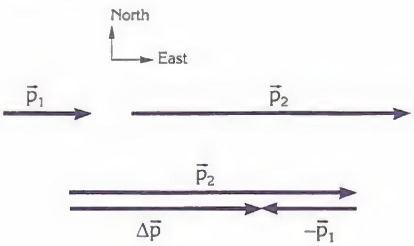
$$p_f = mv_f = (900 \text{ kg})(30 \text{ m/s}) = 2\,700 \text{ kgm/s}$$

Then from the change in momentum

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i$$

$$\Delta p = 2700 \text{ kgm/s} - 900 \text{ kgm/s} = 1800 \text{ kgm/s}$$

In the diagram below, the change in momentum vector is shown, this is in the same direction as the motion of the car, due east.



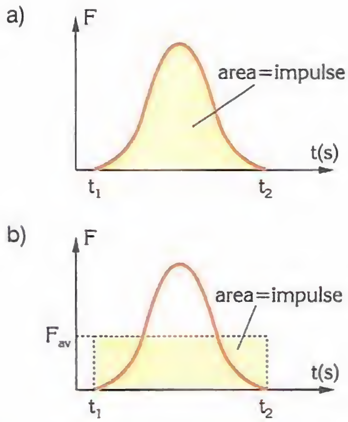



**Figure 7.3** a) A force acting on an object may change in time. The area under the force-time graph gives the impulse. The average force, $\vec{F}_{av}$ gives the same impulse as the actual varying force.

## 7.2 IMPULSE

The equation relating force and change in momentum can be rearranged as follows.

$$\vec{F}\Delta t = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_i$$

This expression is called the **impulse – momentum** theorem. The term, FΔt in this equation is called impulse and labeled I.

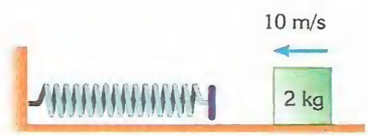
$$\vec{I} = \vec{F}\cdot\Delta t$$

According to the first equation, it is equal to the change in momentum. That is, the impulse, $\vec{I} = \vec{F}\Delta t$ given to an object by a force $\vec{F}$ causes a change in its momentum, $\Delta\vec{p}$. As the equation indicates, impulse is a vector quantity.

During the interaction of objects, for example in collisions, the force exerted by the objects upon each other is not constant. It varies over a time interval which is very short. The force – time graph of a collision is shown in Figure 7.3 a. The area under the force – time curve gives the impulse. Since it is difficult to find this area, it is necessary to define an average force, $\vec{F}_{av}$ as in Figure 7. 3 b. This force is a constant force. The area between this force and the time axis gives the same impulse as the real varying force.

## Example 7.2 — Impulse and average force

A 2-kg object strikes a spring on a frictionless horizontal plane with a velocity of 10 m/s, as shown in the figure, and returns with the same velocity. If the object and the spring interact for $\Delta t = 2$ s, find




a) the change in the momentum of the object,

b) the impulse applied by the spring upon the object.

c) the average force applied by the spring upon the object.

**Solution**

a) The change in momentum of the object is the difference between its momentum just before it strikes the spring and its momentum just after it strikes the spring.

$$\Delta\vec{p} = \Delta\vec{p}_f - \Delta\vec{p}_i$$

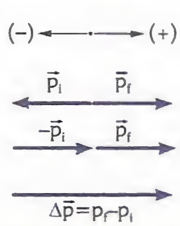
The directions of the momentum vectors $\vec{p}_f$ and $\vec{p}_i$ are opposite each other. Choosing the positive and negative directions, as shown in the diagram;

$$\Delta p = p_f - (-p_i)$$

$$\Delta p = mv_f - (-m\,v_i)$$

$$\Delta p = (2\text{ kg})(10\text{ m/s}) + (2\text{ kg})(10\text{ m/s})$$

$$\Delta p = 40\text{ kgm/s}$$




b) The impulse applied by the spring to the object is equal to the momentum change that was calculated in part a), that is;

$$I = 40\text{ kg}\cdot\text{m/s}$$

c) The impulse;

$$\vec{I} = \vec{F}_{net}\cdot\Delta t$$

For the average force $F_{av}$ will be used here

$$40\text{ kgm/s} = F_{av}(2\text{ s}) \qquad \text{thus} \qquad F_{av} = 20\text{ N}$$

## Example 7.3

Impulse

Why do we bend our knees when we jump down from a higher position?

**Solution**

According to the formula;

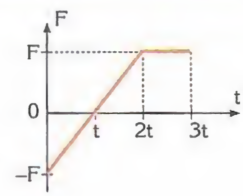
$$F \cdot \Delta t = \Delta P$$

Momentum is constant so it can't be changed. In order to reduce the force F, only the time $\Delta t$ can be increased by bending our knees. The force F decreases and thus, we protect our bones from breaking.

## Example 7.4

Force-time graph

An object at rest experiences a force. Its force versus time graph is shown in the figure. In which time intervals has the momentum increased?




**Solution**

The area under the force-time graph gives the change in momentum of the object or the impulse acting on the object. The impulse acting on the object in the time interval 0-t can be found from the area under the graph,

$$\text{Area} = I_1 = \Delta p_1 = \frac{-F \cdot t}{2}$$

In this time interval, the impulse applied to the object increased its momentum in the negative direction.

The impulse applied in the time interval t-2t is;

$$\text{Area} = I_2 = \Delta p_2 = \frac{F \cdot (2t - t)}{2} = \frac{F \cdot t}{2}$$

In this time interval, the impulse decreased the momentum and at the moment 2t the momentum becomes zero.

The impulse applied in the time interval 2t-3t is

$$\text{Area} = I_3 = \Delta p_3 = F \cdot (3t - 2t) = Ft$$

The impulse in this time interval increased the momentum of the object in the positive direction. Therefore, the momentum is increased in intervals 0-t and 2t-3t.

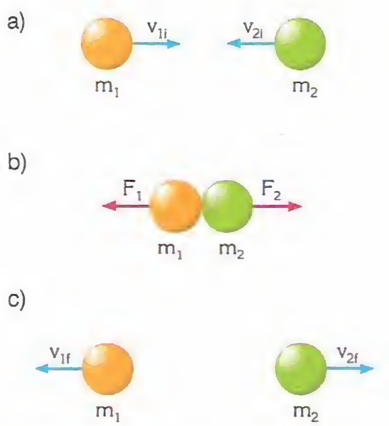



**Figure 7.4** a) Two particles move towards each other, b) during the collision, they exert equal and opposite forces upon each other, c) due to the impulses on the particles, they experience a change in velocity, so a change in momentum.

## 7.3 CONSERVATION OF MOMENTUM

Momentum is an important concept because it is conserved under certain circumstances. In this section the second and third laws of motion will lead to the law of conservation of momentum.

Consider the collision of two particles such as balls, molecules, or any other two objects, as shown in Figure 7.4 a. During the collision the third law of motion describes the forces which the particles exert upon each other. These are equal in magnitude and opposite in direction, as shown in Figure 7.4 b. In the previous section it was seen that the forces during a collision vary in time. Thus average force, $F_{av}$ must be used in applying the laws of motion. From the third law of motion

$$\vec{F}_1 = -\vec{F}_2 \quad (F_1 \text{ and } F_2 \text{ are average forces})$$

Multiplying both sides of this equation with the interaction time $\Delta t$ of collision, the impulse given by the particles to each other is found.

$$\vec{F}_1 \Delta t = -\vec{F}_2 \Delta t$$

Then, from the impulse-momentum equation, we can write

$$\vec{F}_1 \Delta t = \Delta \vec{p}_1, \quad \vec{F}_2 \Delta t = \Delta \vec{p}_2$$

If we substitute the right sides of these two equations into the previous equation for the impulses and then re-arrange the equations,

$$\Delta \vec{p}_1 = -\Delta \vec{p}_2$$

$$\Delta \vec{p}_1 + \Delta \vec{p}_2 = 0$$

$$(\vec{p}_{1f} - \vec{p}_{1i}) + (\vec{p}_{2f} - \vec{p}_{2i}) = 0$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

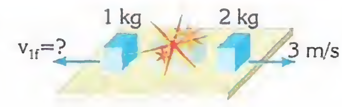
$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

$$\text{Total momentum before collision} = \text{Total momentum after collision}$$

The law of **conservation of momentum** states that, in the absence of external forces, such as gravitational force or friction, when two or more objects collide with each other the total momentum before the collision is equal to the total momentum after the collision.

### Example 7.5 — Conservation of momentum

A 3-kg explosive object, which is at rest on a smooth plane fragments into two pieces of masses 1 kg and 2 kg after an internal explosion. If the 2-kg piece moves due east at a speed of 3 m/s, what is the velocity of the other piece?




**Solution**

The object is at rest and the initial momentum is zero. Since the momentum is conserved.

$$\Sigma\vec{p}_i = \Sigma\vec{p}_f$$

$$0 = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

$$0 = (1\ \text{kg})v_{1f} + (2\ \text{kg})(3\ \text{m/s})$$

$$v_{1f} = -6\ \text{m/s}$$

The 1 kg object is moving to the west at 6 m/s.

## 7.4 COLLISIONS

### a. Inelastic Collisions in One Dimension

In the previous section, it was shown that the total momentum is always conserved for any type of collision. However, total kinetic energy is generally not conserved in collisions. This is because the colliding objects exert great forces upon each other, causing them to undergo deformation and lose some of their kinetic energy. This lost energy is converted into thermal energy and internal elastic potential energy.

Thus, an **inelastic collision** is one in which momentum is conserved but kinetic energy is not.

For example, during the collision of two cars a portion of their kinetic energy is lost as work done to deform the cars.

In this section we will study a completely inelastic collision in which colliding objects stick together after collision.

Consider that two objects of masses $m_1$ and $m_2$ are moving in opposite directions at velocities $v_1$ and $v_2$, as shown in Figure 7.5.a. Also assume that they undergo a head-on collision. That is, they move along the same line before and after the collision. If they stick together and move off together with a common velocity $v_c$ after the collision, as shown in Figure 7.5.b, it can be seen that only momentum will be conserved.

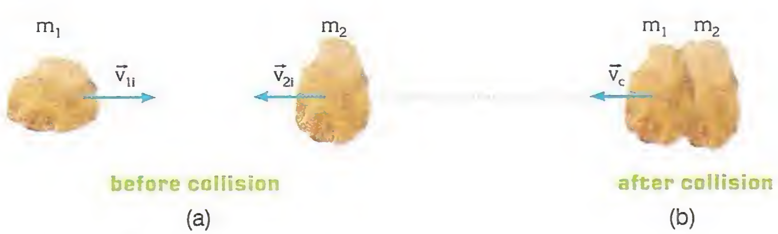



**Figure 7.5** Schematic representation of a completely inelastic collision between two particles.

If the law of conservation of momentum is applied to this collision,

$$\Sigma\vec{p}_i = \Sigma\vec{p}_f$$

$$p_1 + p_2 = p_c$$

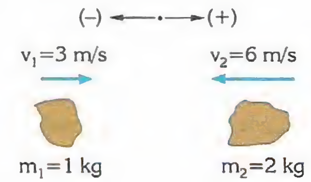
where $p_1$ and $p_2$ are the moment a of the particles before collision and $p_c$ is the common momentum after collision. Thus

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{v}_c$$

In this equation, since velocity is a vector quantity, the signs of the velocities representing directions must be taken into account. In solving problems, first positive and negative directions must be established, and then the values of velocities must be substituted into the equation along with their signs.

## Example 7.6 — Inelastic collisions in one dimension

Two objects of masses $m_1=1$ kg and $m_2=2$ kg undergo a head-on collision at speeds $v_1=3$ m/s and $v_2=6$ m/s, as shown in the figure. If the collision is inelastic




a) what is the common velocity of the composite object after the collision?

b) what is the amount of kinetic energy that the system lost due to the collision?

### Solution

a) Only momentum is conserved in this collision.

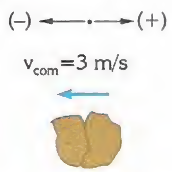
$$\Sigma\vec{p}_i = \Sigma\vec{p}_f$$

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{v}_c$$

$$(1\text{ kg})(3\text{ m/s}) - (2\text{ kg})(6\text{ m/s}) = (1\text{ kg} + 2\text{ kg})v_c$$

$$-9\text{ kgm/s} = 3v_c$$

$$v_c = -3\text{ m/s}$$




The negative sign of the final common velocity indicates that the composite object moves in the negative direction.

b) The kinetic energy which is lost due to the collision is equal to the difference between the kinetic energy before the collision and the kinetic energy after the collision. Which is

$$\Delta E = \Sigma E_{initial} - \Sigma E_{final}$$

$$\Delta E = \left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\right) - \frac{1}{2}(m_1 + m_2)v_{com}^2$$

$$\Delta E = \left(\frac{1}{2}\cdot 1\cdot 3^2 + \frac{1}{2}\cdot 2\cdot 6^2\right) - \frac{1}{2}(1+2)3^2$$

$$\Delta E = 27\text{ J}$$

This lost energy is used while the objects are being stuck together and some is transferred to heat.

## Example 7.7 — Inelastic collisions in one dimension

A stationary wooden block is struck by a bullet with mass $m=0.5$ kg travelling at a speed of $v=100$ m/s, as shown in the figure. As the bullet enters the block, what is the common velocity of the system on the smooth surface? ($\mu=0$)


### Solution

Since the bullet remained inside the block, this collision is inelastic. In inelastic collisions only momentum is conserved. Therefore,

$$\Sigma\vec{p}_i = \Sigma\vec{p}_f$$

$$m\vec{v} = (m + M)\vec{v}_c$$

$$(0.5\text{ kg})(100\text{ m/s}) = (0.5\text{ kg} + 9.5\text{ kg})v_c$$

$$50\text{ kgm/s} = 10v_c$$

$$v_c = 5\text{ m/s}$$

## b. Elastic Collisions in One Dimension

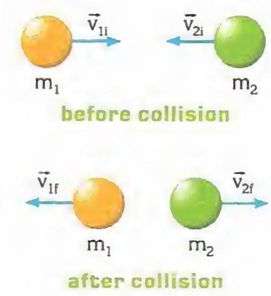



**Figure 7.6** Schematic representation of an elastic head-on collision between two particles.

Collisions in which both momentum and kinetic energy are conserved are called **elastic collisions.** Assume that two objects undergo a perfectly elastic head on collision, as shown in Figure 7.6. If the laws of conservation of momentum and kinetic energy are applied to the objects, labeling their velocities $v_{1i}$ and $v_{2i}$ before the collision and $v_{1f}$ and $v_{2f}$ after the collision,

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f} \qquad \text{(conservation of momentum)}$$

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2 \qquad \text{(conservation of kinetic energy)}$$

In order to simplify the applications of these equations, they can be rearranged as follows: Gather the terms related to the first object on the left side of the equations and the terms related to the second object on the right side. Thus, for the law of conservation of momentum

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

$$m_1\vec{v}_{1i} - m_1\vec{v}_{1f} = m_2\vec{v}_{2f} - m_2\vec{v}_{2i}$$

$$m_1(\vec{v}_{1i} - \vec{v}_{1f}) = m_2(\vec{v}_{2f} - \vec{v}_{2i}) \qquad (1)$$

Then for the law of conservation of kinetic energy

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

$$m_1v_{1i}^2 - m_1v_{1f}^2 = m_2v_{2f}^2 - m_2v_{2i}^2$$

$$m_1(v_{1i}^2 - u_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2)$$

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) \qquad (2)$$

Taking the ratio of the simplified equations (1) and (2) an expression for the velocities of the objects can be found

$$\frac{m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i})}{m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i})}$$

Using the equation relating the velocities of the objects before and after the collision

$$\vec{v}_{1i} + \vec{v}_{1f} = \vec{v}_{2i} + \vec{v}_{2f}$$

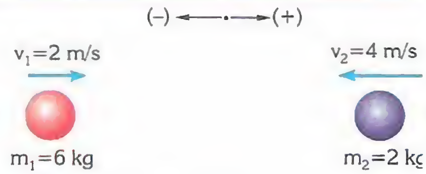
This equation in combination with that for the conservation of momentum will be applied to objects undergoing perfectly elastic collisions.

As in the applications of inelastic collisions in one dimension, first the signs for the negative and positive directions must be established, then the values of the velocities must be substituted into the equations with these signs.

# Example 7.8



Two objects of masses $m_1=6$ kg and $m_2=2$ kg approach each other with speeds $v_1=2$ m/s and $v_2=4$ m/s and undergo a head-on elastic collision. Find the velocities of the objects after the collision.




**Solution**

Since the collision is elastic, both energy and momentum are conserved. Let us denote the velocity of the object of mass $m_1$ after the collision as $\vec{v}_{1f}$ and the velocity of the object of mass $m_2$ as $\vec{v}_{2f}$. Hence,

$$\Sigma\vec{p}_i = \Sigma\vec{p}_f$$

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

$$(6\text{ kg})(2\text{ m/s}) - (2\text{ kg})(4\text{ m/s}) = (6\text{ kg})v_{1f} + (2\text{ kg})v_{2f}$$

$$(6\text{ kg})v_{1f} + (2\text{ kg})v_{2f} = 4\text{ kgm/s} \qquad (1)$$

Instead of the equation for the conservation of energy, if the equation for the conservation of velocities is used

$$\vec{v}_{1i} + \vec{v}_{1f} = \vec{v}_{2i} + \vec{v}_{2f}$$

$$2\text{ m/s} + v_{1f} = -4\text{ m/s} + v_{2f}$$

$$v_{2f} - v_{1f} = 6\text{ m/s} \qquad (2)$$

substituting equation (2) into equation (1),

$$\begin{array}{rl} & 6v_{1f} + 2v_{2f} = 4 \\ + & -2/\ \ v_{2f} - v_{1f} = 6 \\ \hline & 6v_{1f} + 2v_{2f} = 4 \\ + & -2v_{2f} + 2v_{1f} = -12 \\ \hline & 8v_{1f} = -8 \\ & v_{1f} = -1\text{ m/s} \end{array}$$

Placing this value in the equation; $v_{2f} - v_{1f} = 6$

$$v_{2f} = 5\text{ m/s}$$

Thus, mass $m_1$ will move with a speed of 1 m/s in the negative direction, and mass $m_2$ will move with a speed of 5 m/s in the positive direction.

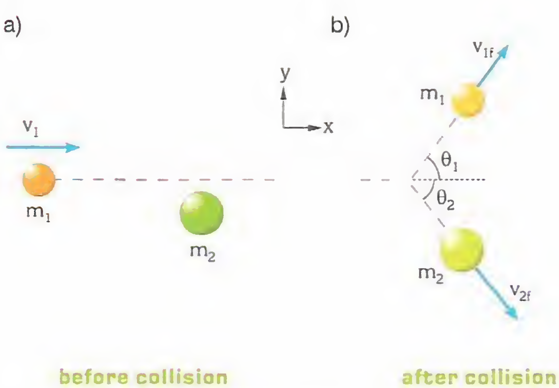



**Figure 7.7** a) Before an elastic collision and b) after an elastic collision in two dimensions

## c. Collisions in Two Dimensions

In the last two sections, head-on collisions have been analysed. That is, collisions that take place in one dimension. In this section, collisions that take place in two dimensions will be analysed. A good example of such a collision is the collision of two billiard balls moving in different directions after their collision, as shown in Figure 7.7. Here

Ball 1 is heading along the x axis towards ball 2, which is initially at rest. Then ball 1 of mass $m_1$, strikes ball 2 of mass $m_2$ and they go off at angles $\theta_1$ and $\theta_2$, respectively. The velocities of the balls after their collision are labeled $v_{1f}$ and $v_{2f}$, respectively.

Assuming that they undergo a perfectly elastic collision, the law of conservation of momentum can be applied. Since momentum is a vector quantity and the collision takes place in two dimensions, the equation will be applied along the x direction and along the y direction separately

$$\Sigma\vec{p}_{ix} = \Sigma\vec{p}_{fx} \text{ and } \Sigma\vec{p}_{iy} = \Sigma\vec{p}_{fy}$$

$$m_1\vec{v}_{1ix} + m_2\vec{v}_{2ix} = m_1\vec{v}_{1fx} + m_2\vec{v}_{2fx} \text{ (along the x direction)}$$

$$m_1\vec{v}_{1iy} + m_2\vec{v}_{2iy} = m_1\vec{v}_{1fy} + m_2\vec{v}_{2fy} \text{ (along the y direction)}$$

If the collision is perfectly inelastic in two dimensions the conservation of momentum can be applied as follows

$$\Sigma\vec{p}_i = \Sigma\vec{p}_f \quad \Rightarrow \quad \vec{p}_1 + \vec{p}_2 = \vec{p}_c$$

$$\Sigma\vec{p}_{ix} = \Sigma\vec{p}_{cx}, \quad \Sigma\vec{p}_{iy} = \Sigma\vec{p}_{cy}$$

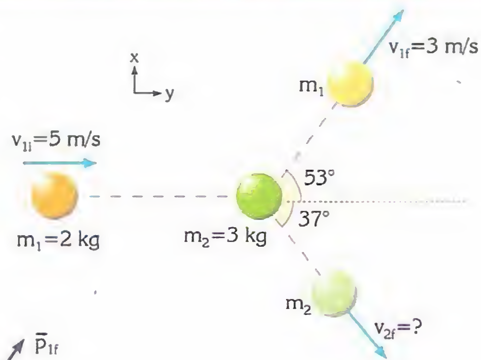
$$m_1\vec{v}_{1x} + m_2\vec{v}_{2x} = (m_1 + m_1)\vec{v}_{cx}$$

$$m_1\vec{v}_{1y} + m_2\vec{v}_{2y} = (m_1 + m_1)\vec{v}_{cy}$$

where $v_c$ refers to the common velocity after collision.

## Example 7.9 Elastic collision in two dimensions

A 2-kg object moving with a velocity of 5 m/s collides with a stationary object of mass $m_2$=3 kg. The objects then move off in different directions, as shown in the figure. If the velocity of mass $m_1$ after the collision is $v_{1f}$=3 m/s, what is the velocity of mass $m_2$ after the collision?




### Solution 1

Since the collision is not head-on, the conservation of momentum in the x and y directions can be analysed separately. The conservation of momentum along the x axis is

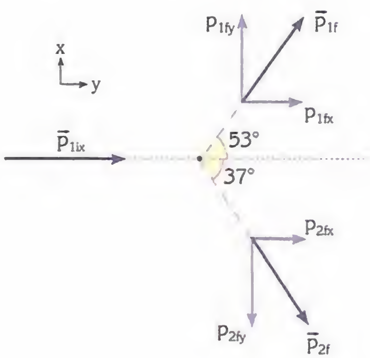
$$\Sigma\vec{p}_{i,\,x} = \Sigma\vec{p}_{f,\,x}$$

$$\vec{p}_{1ix} + \vec{p}_{2ix} = \vec{p}_{1fx} + \vec{p}_{2fx}$$

$$m_1v_{1i} + m_2 0 = m_1v_{1f}\cos 53° + m_2v_{2f}\cos 37°$$

$$(2\text{ kg})(5\text{ m/s}) = (2\text{ kg})(3\text{ m/s})0.6 + (3\text{ kg})v_{2f}0.8$$

$$v_{2f} = \frac{8}{3}\text{ m/s}$$



### Solution 2

The velocity of mass $m_2$, $v_{2f}$ can be obtained after the collision by applying the equation of conservation of momentum along the y axis.

$$\Sigma\vec{p}_{i,\,y} = \Sigma\vec{p}_{f,\,y}$$

$$\vec{p}_{1iy} + \vec{p}_{2iy} = \vec{p}_{1fy} + \vec{p}_{2fy}$$

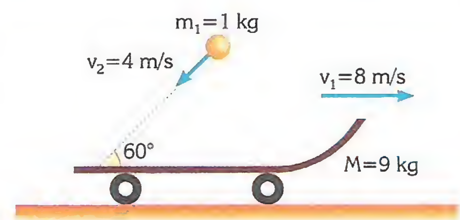
$$m_1 0 + m_2 0 = m_1 v_{1f} \cos 53° + m_2 v_{2f} \cos 37°$$

$$0 = (2\text{ kg})(3\text{ m/s})0.8 - (3\text{ kg})v_{2f}0.6$$

$$v_{2f} = \frac{8}{3}\text{ m/s}$$

## Example 7.10 Inelastic collision in two dimensions

While a car of mass $M=9$ kg is moving with a velocity of 8 m/s in the horizontal, direction some beeswax of mass $m=1$ kg is thrown towards the car at a velocity of 4 m/s, making an angle of 60° with the horizontal, as shown in the figure. If they stick and move off together, what is their common velocity? (Take cos 60° = 0.5)




### Solution

The beeswax has a momentum both in the x direction and in the y direction before it strikes the car. Its momentum in the y direction is not conserved. However, the momentum in the x direction is conserved.

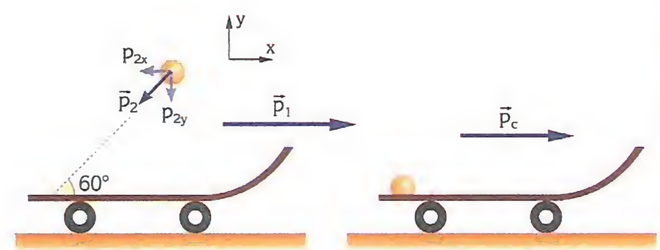
$$\Sigma\vec{p}_{i,x} = \Sigma\vec{p}_{f,x}$$

$$\vec{p}_{1x} + \vec{p}_{2x} = \vec{p}_c$$

$$M v_1 - m v_2 \cos 60° = (M+m)v_c$$

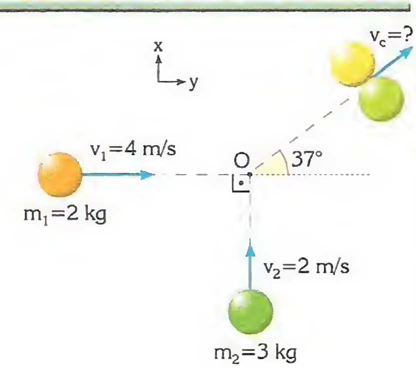
$(9\text{ kg})(8\text{ m/s}) - (1\text{ kg})(4\text{ m/s})0.5 = (9\text{ kg}+1\text{ kg})v_c$ thus

$v_c = 7$ m/s




## Example 7.11 Collision in two dimensions

Two objects of masses $m_1=2$ kg and $m_2=3$ kg move towards each other with velocities of $v_1=4$ m/s and $v_2=2$ m/s, as shown in the figure. They collide at point O and stick together. The composite object moves along a path making an angle of 37° to the x axis. What is the velocity of the composite object after the collision?




**Solution 1**

Since the collision is inelastic in two dimensions the conservation of momentum along the x-axis and y-axis can be analysed separately. Stating the conservation of momentum along the x axis

$$\Sigma\vec{p}_{ix} = \vec{p}_{cx}$$

$$\vec{p}_{1x} + \vec{p}_{2x} = \vec{p}_{cx}$$

$$m_1v_1 + m_2 0 = (m_1+m_2)v_{cx}$$

$$(2\text{ kg})(4\text{ m/s}) = (2\text{ kg} + 3\text{ kg})v_c\cos 37°$$

$$v_c = 2\text{ m/s}$$

**Solution 2**

The problem can also be solved by using the conservation of momentum along the y axis.

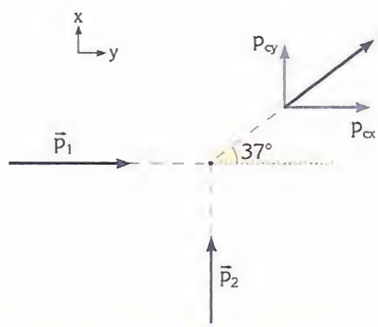
$$\Sigma\vec{p}_{iy} = \vec{p}_{cy}$$

$$\vec{p}_{1y} + \vec{p}_{2y} = \vec{p}_{cy}$$

$$m_1 \,.\, 0 + m_2v_2 = (m_1+m_2)v_{cy}$$

$$(3\text{ kg})(2\text{ m/s}) = (2\text{ kg} + 3\text{ kg})v_c\sin 37°$$

$$v_c = 2\text{ m/s}$$




**Solution 3**

The problem can also be solved using the vector addition of the momenta of the objects. Using the conservation of momentum

$$\Sigma\vec{p}_i = \Sigma\vec{p}_f$$

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_c$$

$$\vec{p}_1 = 8\text{ kgm/s}$$

$$\vec{p}_2 = 6\text{ kgm/s}$$

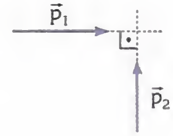
$$\vec{p}_c^{\,2} = \vec{p}_1^{\,2} + \vec{p}_2^{\,2}$$

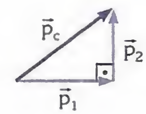
$$\vec{p}_c = 64 + 36$$

$$\vec{p}_c = 10\text{ kgm/s} = m_cv_c$$

$$10\text{ kgm/s} = (5\text{ kg})v_c$$

$$v_c = 2\text{ m/s}$$




## 7.5 ROCKETS

When an inflated balloon is released into the air with its end open, it is observed that the balloon moves in the opposite direction to that of the air propelled out of the balloon (Figure 7.8).


**Figure 7.8** An inflated balloon, when released, moves in the opposite direction to that of the air propelled from it.

The mechanism by which rockets move is similar to that of a balloon. While a rocket moves forwards in space, it burns fuel, which is a part of its mass, and ejects it as a gas at a very high velocity. It is significant that the effective forces here are all internal forces of the system. Momentum is conserved here, but energy is not conserved.

At a given time the mass of the rocket and its fuel can be represented as M, a mass of fuel, Δm will be ejected in a very short time interval Δt, the velocity of the rocket is represented by v. The initial momentum of the system consisting of rocket + fuel will be represented by (M+Δm)v (Figure 7.9). After a short time interval Δt,


$p_i=(M+\Delta m)v$ $\quad$ $p_f=M(v+\Delta v)+\Delta m(v-v_f)$

**Figure 7.9** The law of conservation of momentum is applied when describing the method by which rockets move.

the rocket ejects fuel of mass $\Delta m$ and the velocity of the rocket becomes $v + \Delta v$ (Figure 7.9).

If the velocity of the fuel is $v_{fuel}$ relative to the rocket, it will be $v - v_{fuel}$ relative to an observer outside the system. Equating the total momentum of the system before and after the fuel is ejected

$$(M + \Delta m)v = M(v + \Delta v) + \Delta m(v - v_{fuel})$$

Rearranging this equation

$$M\Delta v = \Delta m v_{fuel}$$

It is clear that the change in velocity of a rocket depends on the equation below;

$$\Delta v = \frac{\Delta m}{M} v_{fuel}$$

If both sides of this equation is divided by $\Delta t$, and rearranged

$$\frac{M\Delta v}{\Delta t} = \frac{\Delta m}{\Delta t} v_{fuel}$$

since $\frac{\Delta v}{\Delta t} = a$, the equation becomes

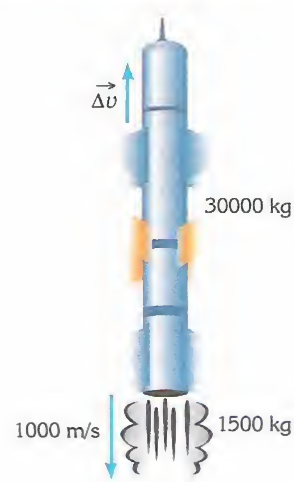
$$Ma = \frac{\Delta m}{\Delta t} \cdot v_{fuel}$$

Ma in this equation is the force (thrust) exerted on the rocket by the ejected combusted fuel.

Thus, the thrust increases as the combusted fuel's velocity ($v_{fuel}$) increases and as the rate of change of mass (burn rate) increases.

## Example 7.12 — The speed of a rocket

If a stationary rocket which has a mass of 30000 kg ejects combusted fuel of mass 1500 kg in a short time interval $\Delta t$, at a velocity of 1000 m/s, what will its velocity be after the time interval $\Delta t$?




### Solution

The change in velocity of the rocket is found, using the equation;

$$\Delta v = \frac{\Delta m}{M} \cdot v_{fuel}$$

$$\Delta v = \frac{1500}{30000} \cdot 1000 \quad \text{thus} \quad \Delta v = 50\ \text{m/s}$$

Since the initial velocity of the rocket is zero, this change in velocity is also the velocity that the rocket gains.

# Summary

The momentum of an object of mass m and velocity v is given as;

$$\vec{p} = m\vec{v}$$

When a force is applied to an object, the object will accelerate, the velocity and momentum will hence change. The resultant force;

$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t}$$

Rearranging the equation;

$$\vec{F}\Delta t = \Delta\vec{p}$$

In the equation above, the expression FΔt is described as the impulse applied to the body and is denoted as I. Hence, the impulse is equal to the product of the applied force multiplied by the application time.

$$\vec{I} = \vec{F}\Delta t$$

Impulse, force, momentum and velocity are vector quantities. The direction of momentum is the same as the direction of velocity.

The area under the force-time (F - t) graph of an object gives the impulse applied to the object or the change in momentum of the object.

The law of **conservation of momentum** states that, in the absence of external forces, such as gravitational force or friction, when two or more objects collide with each other the total momentum before the collision is equal to the total momentum after collision. That is;

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

$$\text{Total momentum before collision} = \text{Total momentum after collision}$$

There are two types of collisions: Elastic collisions and inelastic collisions.

**Inelastic collision** is one in which momentum is conserved but kinetic energy is not.

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{v}_c$$

Collision in which both momentum and kinetic energy are conserved is called **elastic collision.**

According to the law of conservation of momentum;

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

According to the law of conservation of kinetic energy;

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

Then we obtain an equation relating the velocities of the objects before and after collision as

$$\vec{v}_{1i} + \vec{v}_{1f} = \vec{v}_{2i} + \vec{v}_{2f}$$

If collisions occur in a direction along a line which joins the centres of mass of colliding objects, the collision is said to be head-on. If collisions occur in two dimensions, the collision is said to be a two dimensional collision. The conservation of momentum in a two dimensional collision is given by

$$\Sigma\vec{p}_{ix} = \Sigma\vec{p}_{fx} \quad \text{and} \quad \Sigma\vec{p}_{iy} = \Sigma\vec{p}_{fy}$$

by considering them separately along the x and y axes.

The conservation of momentum is also used to explain the motion of rockets. If a rocket of mass M ejects an amount of fuel Δm at a velocity $v_f$ the change in velocity of the rocket will be equal to;

$$\Delta v = \frac{\Delta m}{M}v_f$$

# QUESTIONS AND PROBLEMS

## 7.1 Momentum

1. What does the momentum of an object depend upon?

2. What is the momentum of an 18 g bullet which moves at a velocity of 250 m/s?

3. What is the momentum of an electron, having a mass of $9.1 \cdot 10^{-31}$ kg, which moves at a velocity of 200 km/s?

4. The velocity of a 1800-kg lorry which has the same momentum as a 900-kg car, is 36 km/h. What is the velocity of the car?

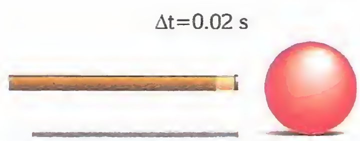
5. A 0.2 kg ball is thrown vertically upwards with an initial velocity of 20 m/s
   a) What is its momentum when it is at its maximum height?
   b) What is its momentum when it is halfway to its maximum height?

6. What is the change in the momentum of a 1200 kg car when its velocity decreases from 180 km/h to 36 km/h?

## 7.2 Impulse

7. Which equation is used to calculate impulse?

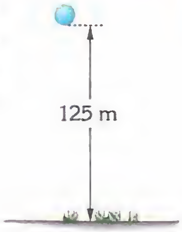
8. What is the impulse applied by a boxer to his rival when he strikes him with a force of 100 N in 0.05 s?

9. 

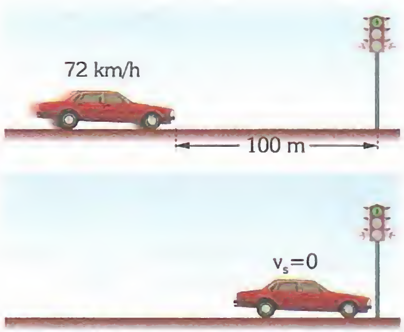

   An average force of 40 N which is parallel to the ground is applied to a billiard ball in 0.02 s.
   a) What is the impulse applied to the ball?
   b) What is the velocity given to the ball if its mass is 200 g?

10. A basketball player is bouncing on a flat surface. The time of interaction of the ball with the surface is 1/600 s and the impulse that the surface applies to the ball is 3 Ns. What is the average force applied by the surface to the ball?

**11.** A 2kg object is released from a height of 125 m, strikes the ground and then stops without bouncing. Calculate the impulse applied by the ground upon the object.




**12.**



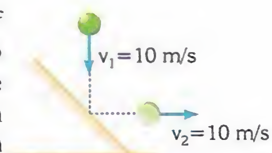

A driver travelling at a velocity of 72 km/h sees a red light at a distance of 100 m. He breaks and stops near the traffic lights, as shown in the figure. If the mass of the car including the driver is 800 kg

a) what impulse is applied to the car?

b) If the driver has a mass of 60 kg, what impulse is applied to him?

c) What is the average stopping force applied to the car by the brakes?

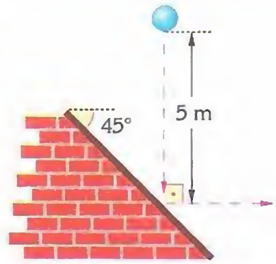
**13.** If the velocity of 10 m/s of a ball, with a mass 0.5 kg, is the same just before and just after a collision with a wall, as shown in the figure



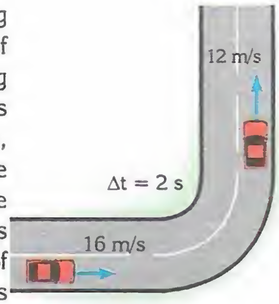

a) what is the change in the momentum of the ball?

b) if the interaction time between the wall and the ball is 0.2 s, find the average force exerted by the wall on the ball.

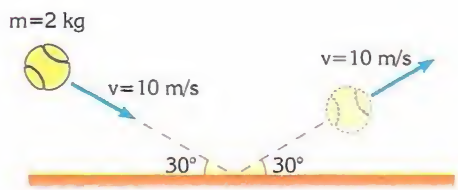
**14.** A 1 kg object released from a height of 5 m strikes a hard plane making an angle of 45° with the horizontal, as shown in the figure. After the object strikes the plane it changes direction so that its new direction makes an angle of 90° with its initial direction. What is the impulse given by the plane to the object? (Assume that the magnitude of the velocity of the ball just before and just after the collision is the same.)




**15.** A 1000 kg car is moving at a constant velocity of 16 m/s, and while driving round a curve it changes its direction by 90° in 2 s, under the effect of the friction force on the road. If the car continues travelling at a velocity of 12 m/s after it completes the turn, what is the average friction force applied by the ground to the car?




**16.**

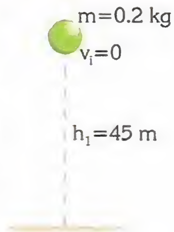



A tennis ball of mass m=2 kg strikes the ground with a velocity of 10 m/s, making an angle of 30° with the horizontal, and bounces off, as shown in the figure. Find the impulse delivered by the ground to the ball.

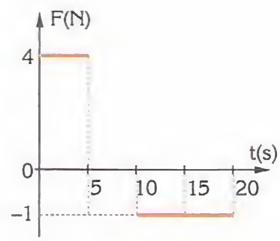
**17.** When a ball is released from a height of 45 m, it rises to a maximum height of 20 m after it makes a collision with the ground. If the interaction time is 0.1 s, find

a) the impulse given by the ground to the ball.

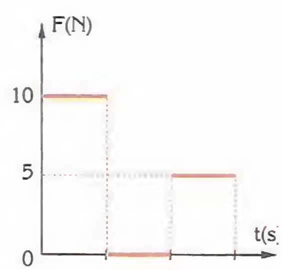
b) the average force exerted by the ground on the ball.




**18.** The force-time graph of an object is shown in the figure. What is the change in momentum of the object between t = 0 and t = 20 s?




**19.** The graph of force versus time for a 2 kg object with an initial velocity of $v_0=10$ m/s is as shown in the figure. What is the momentum of the object at the end of 6 seconds?




## 7.3 Conservation of Momentum

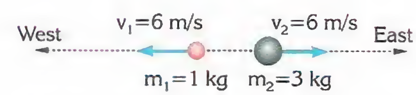
**20.** A 6 kg chemistry tube, which is at rest on a smooth plane, fragments into two pieces of masses 2 kg and 4 kg after an internal explosion, as shown in the figure. If the 2 kg piece moves due west at a velocity of 10 m/s after the explosion, what is the velocity of the other piece ?


**21.** A 30 kg child at rest with roller skater throws a 1 kg ball at a velocity of 6 m/s to his friend horizontally?

a) What is the impulse acting on the child?

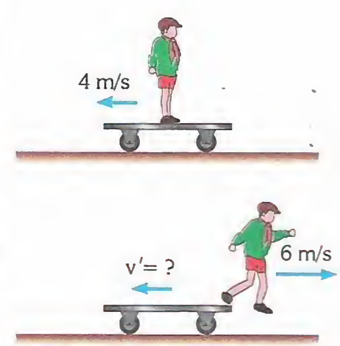
b) Neglecting the friction on the ground, at what velocity and in which direction does the child move?

**22.**

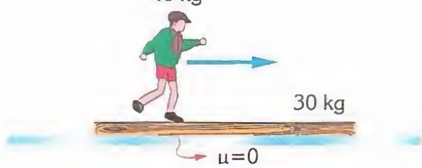



A 4 kg mass fragments into two pieces as a result of an internal explosion. If the pieces move as shown in the figure, what is the velocity of the object before the explosion?

**23.** A 30 kg child is standing on a 25-kg car which is moving at a velocity of 4 m/s relative to the ground, as shown in the figure. If the child jumps off the car with a velocity of 6 m/s relative to the ground, in the opposite direction to that of the car, what will the final velocity of the car relative to the ground be?



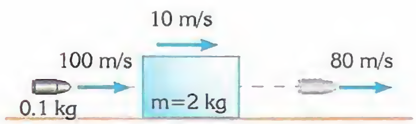
24.




A long wooden beam with a mass of 30 kg is at rest on a smooth surface. A 40 kg child who is initially stationary starts walking on the beam and reaches a velocity of 1.2 m/s relative to the ground in 2 s.

a) What is the velocity of the beam?
b) What is the magnitude of the force applied by the child on the beam?

25.



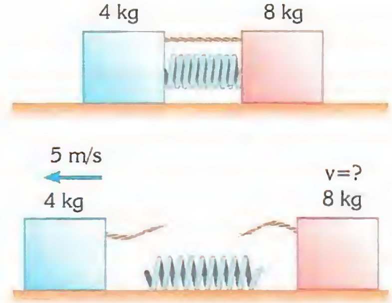

While a 2 kg object is moving at a velocity of 10 m/s, a 0.1 kg bullet is fired into the object at a velocity of 100 m/s and leaves it at a velocity of 80 m/s, as shown in the figure. What is the final velocity of the object? (Neglect any friction and energy transferred into heat.)

26. A hunter whose mass is 75 kg fires a 5 kg rifle. A bullet with a mass of 40 g leaves the rifle with a velocity of 500 m/s. Find the recoil velocity of the rifle for the conditions below


a) if the hunter holds the rifle loosely.

b) if the hunter holds the rifle firmly and presses it against his shoulder.

27.




A spring is placed between two objects of masses 4 kg and 8 kg on a frictionless plane. It is totally compressed by the objects, which are then tied together with a piece of cord, as shown in the figure. After the cord is cut, if the 4 kg object moves towards the left with a velocity of 5 m/s

a) what is the velocity of the 8 kg object?

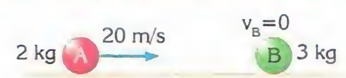
b) what are the kinetic energies of the objects?

c) how much energy is initially stored in the spring?

## 7.4 Collisions

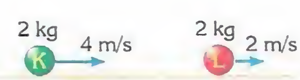
### a) Inelastic Collisions in One Dimension

28.



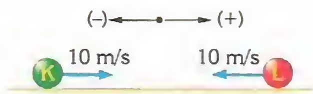

A 2 kg ball moving with a velocity of 20 m/s on a frictionless horizontal surface undergoes a head–on inelastic collision with another object, as shown in the figure. If they move off together after the collision, what is the common velocity of the balls?

29.




Two 2 kg balls moving with velocities of 4 m/s and 2 m/s on a smooth horizontal surface undergo a head – on inelastic collision, as shown in the figure. What is the common velocity of the balls after the collision?

**30.**

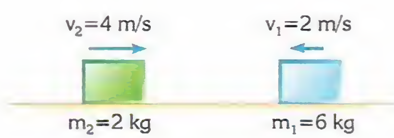



The balls K and L, moving towards each other with velocities of 10 m/s, collide head-on inelastically on a frictionless horizontal surface, as shown in the figure. What is the common velocity of the balls, if;

a) $m_K = m_L = 2$ kg?

b) $m_K = 3$ kg and $m_L = 2$ kg?

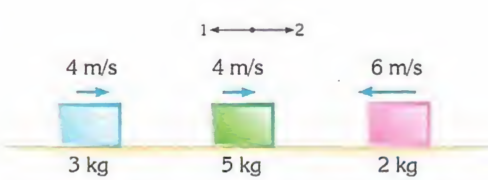
**31**




Two objects which are moving in opposite directions on a smooth horizontal plane undergo an inelastic collision. What is the common velocity of the objects after the collision?

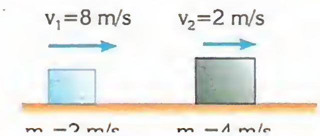
**32.** A 50 g bullet moving at a velocity of 240 m/s strikes and enters a stationary concrete block with a mass of 4950 g. What is the common velocity of the block and the bullet after the collision?

**33.**



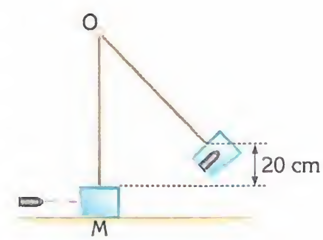

Three objects moving on a smooth horizontal plane, as shown in the figure, collide and stick together at the same time. What is the common velocity of the objects after the collision?

**34.**

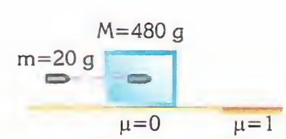



Two masses moving in the same direction on a smooth horizontal plane undergo an inelastic collision and move off together. What is the ratio of the total energy after the collision to the total energy before the collision $\frac{E_f}{E_i}$?

**35.** A 20g bullet is fired into a ballistic pendulum of mass 1980 g, as shown in the figure. If the pendulum's block rises 20 cm as a result, what is the initial velocity of the bullet?




**36.** A 20 g bullet at a velocity of 250 m/s strikes and enters a wooden block of mass m=480 g, which is at rest on a smooth horizontal plane, as shown in the figure. After moving a certain period of time on the smooth plane, the wooden block slides onto a rough surface. If the coefficient of kinetic friction between the surface and the block is 1

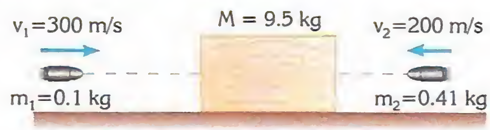



a) at what velocity do the bullet and the block move off together with after the collision?

b) what distance does the block travel along the rough surface?

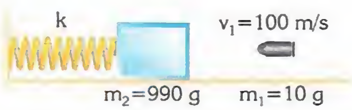
c) what is the impulse applied by the friction force to the wooden block?

**37.**




A stationary wooden block on a smooth surface is struck by two bullets of masses $m_1$=0.1 kg and $m_2$=0·4 kg travelling at velocities of $v_1$=300 m/s and $v_2$=200 m/s in opposite directions, as shown in the figure. If the bullets enter the block, what is the common velocity of the system after the collisions?

**38.**



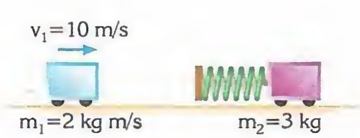

A bullet of mass $m_1$=10 g enters a wooden block of mass $m_2$=990 g, which is fixed to a wall by a spring, as shown in the figure. If the initial velocity of the bullet is 100 m/s and the spring constant is k=100 N/m,

a) what is the common velocity of bullet+block just after the bullet enters the block?

b) How many cm is the maximum compression of the spring?

(Neglect the friction between the block and the surface.)

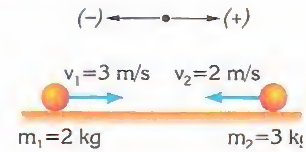
**39.**




A spring with spring constant k=3000 N/m is attached to a car which rests on a smooth horizontal plane. The total mass of the car with the spring is $m_2$=3 kg. How many cm is the spring compressed when another car of mass $m_1$=2 kg strikes it with a velocity of 10 m/s, as shown in the figure?

### b) Elastic Collisions in One Dimension

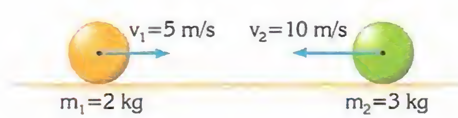
**40.** Two objects of mass $m_1$ and $m_2$ moving at velocities $v_1$ and $v_2$, undergo a head-on elastic collision, as shown in the figure. After the collision, if the object of mass $m_1$ moves at a velocity of $u_1$ and the object of mass $m_2$ moves at a velocity of $u_2$, what are the velocities $\vec{u}_1$ and $\vec{u}_2$?




**41.**


Two identical balls moving at velocities of 4 m/s and –2 m/s, as shown in the figure, undergo a head-on elastic collision. Calculate the final velocities of these balls after the collision.

**42.**

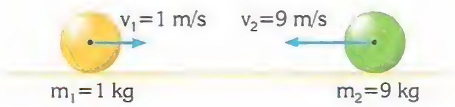



Two balls of masses $m_1$=2 kg, $m_2$=3 kg move towards each other. They have velocities of $v_1$=5 m/s and $v_2$=10 m/s, as shown in the figure. If they undergo a head-on perfectly elastic collision, find the velocities of the objects after the collision.

**43.**

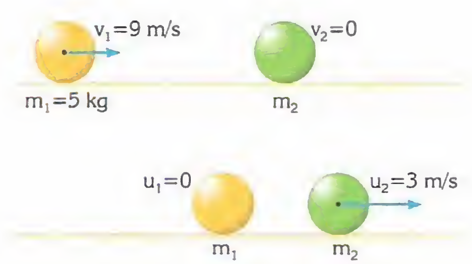

Two balls of $m_1=10$ kg and $m_2=2$ kg moving on a smooth horizontal plane with velocities of 6 m/s and 3 m/s in the same direction collide perfectly elastically. What are the velocities of the objects after the collision?

**44.**



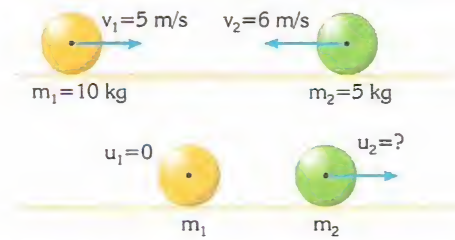

Two balls of masses $m_1=1$ kg and $m_2=9$ kg which are moving towards each other at velocities of $v_1=1$ m/s and $v_2=9$ m/s collide with each other. If the collision is a perfectly elastic head-on collision, what are the velocities of masses $m_1$ and $m_2$ after the collision?

**45.**




An object of mass $m_1=5$ kg moving at a constant velocity of $v_1=9$ m/s undergoes a head-on, perfectly elastic collision with a stationary object of mass $m_2$, as shown in the figure. If the mass $m_2$ starts moving at a velocity of $u_2=3$ m/s after the collision, what is the mass $m_2$?

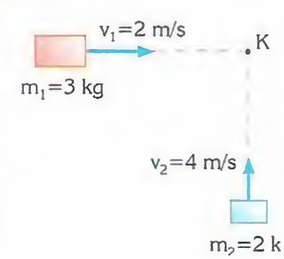
**46.**



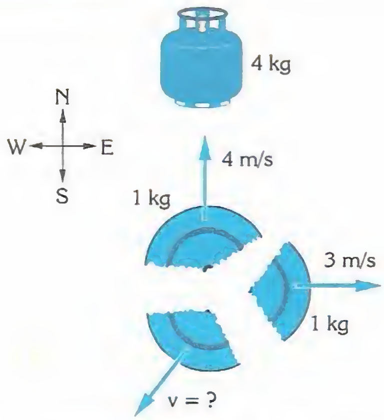

Two balls of masses $m_1=10$ kg and $m_2=5$ kg which are moving at constant velocities of $v_1=5$ m/s and $v_2=6$ m/s collide with each other. If the collision is a perfectly elastic, head-on collision and mass $m_1$ is brought to rest after the collision, what is the velocity $u_2$ of mass $m_2$?

### c) Collision in Two Dimensions

**47.** Two masses moving in different directions, as shown in the figure, collide and stick together at point K. What is their common velocity after the collision? (Neglect any friction effects.)



48.



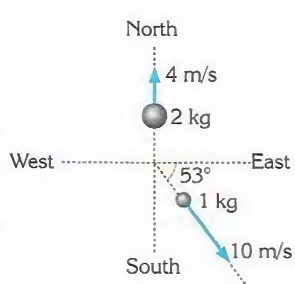

A 4 kg small gas bottle which is at rest on a smooth surface fragments into three pieces as a result of an internal explosion. If the first 1 kg piece moves due east at a velocity of 3 m/s and the second 1 kg piece moves due north at a velocity of 4 m/s, as shown in the figure

a) what is the velocity of the other piece?

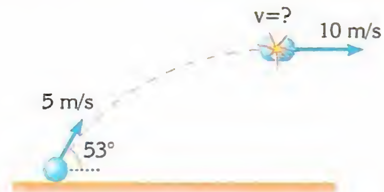
b) how much energy is released after the explosion?

49. An object moving on a frictionless horizontal plane fragments into two pieces as a result of an internal explosion.

The pieces move in the directions shown in the figure after the explosion. What is the velocity of the object before the explosion?




50.

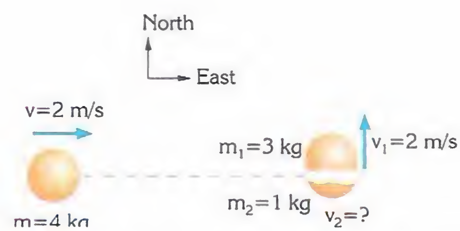



A 6 kg object is thrown with an initial velocity of 5 m/s at an angle of 53° with the horizontal. At its maximum height, the object fragments into two pieces of masses 2 kg and 4 kg, due to an internal explosion. If the small piece moves in the same direction as that of the object before the explosion, with a velocity of 10 m/s, as shown in the figure,

a) what is the velocity of the other piece?

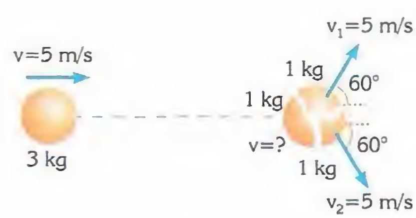
b) what is the distance between the pieces when they strike the ground?

51.



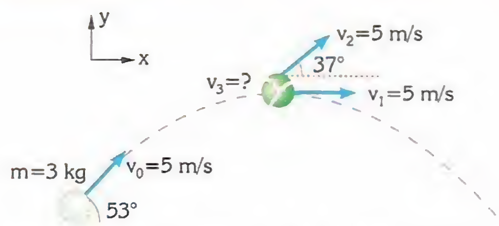

While a mass of m=4 kg moves along a horizontal plane with a velocity v=2 m/s, it is fragmented into two pieces of masses $m_1$=3 kg and $m_2$=1 kg as a result of an internal explosion, as shown in the figure. If the 3 kg piece moves due north with a velocity $v_1$=2 m/s, find the velocity of the other piece.

52.



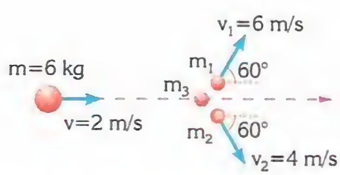

An object of mass 3 kg moving on a smooth horizontal plane with a velocity v=5 m/s fragments into three equal pieces. If two of the pieces move as shown in the figure, with velocities v=5 m/s, what is the velocity of the third piece?

**53.**



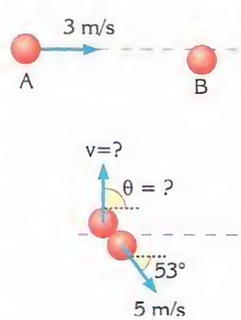

A 3 kg bomb is projected with a velocity of 5 m/s at an angle of 53° with the horizontal. When the bomb is at its maximum height, it explodes and fragments into three equal pieces. The pieces undergo projectile motion, one piece with a horizontal velocity of 5 m/s, the other piece moves outwards with a velocity of 5 m/s at an angle of 37° to the horizontal. What is the velocity of the third piece?

**54.**

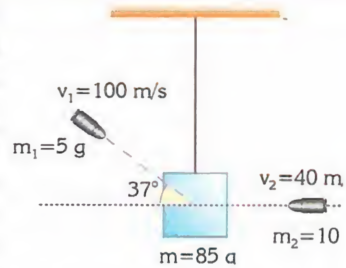



A 6 kg object moving on a smooth horizontal plane at a velocity of 2 m/s fragments into three pieces as a result of an internal explosion. If the piece of mass $m_1=2$ kg moves at a velocity of $v_1=6$ m/s and the piece of mass $m_2=3$ kg moves at a velocity of $v_2=4$ m/s, what is the velocity of the third piece?

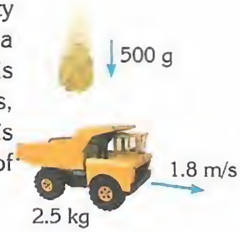
**55.** Ball A which is moving on a smooth plane at a velocity of 3 m/s collides with an identical stationary ball B elastically. Just after the collision ball B moves at a velocity of 5 m/s making an angle of 53° with the incoming direction of ball A, as shown in the figure. What is the velocity of ball A?



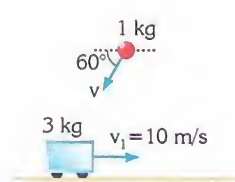

**56.** Two bullets of masses $m_1=5$ g and $m_2=10$ g, are fired into an 85 g suspended wooden block at the same time, as shown in the figure. One bullet has a velocity of 100 m/s and makes an angle of 37° with the horizontal. The other bullet has a velocity of 40 m/s and moves in the horizontal direction. If it takes 0.3 s for the bullets to stop after they enter the block, what is the maximum tension in the cord?



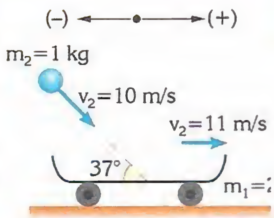

**57.** In a frictionless environment a 500 g piece of window putty falls vertically and sticks on a 2.5 kg toy lorry, which is moving at a velocity of 1.8 m/s, as shown in the figure. What is the change in the velocity of the lorry?



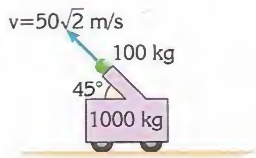

**58.** While a car of mass 3 kg is moving at a velocity of 10 m/s, an object of mass 1 kg collides with it in the direction shown in the figure. If they stick together and move to the right with a common velocity of 5 m/s, at what velocity, v did the object strike the car?



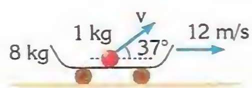
59. While a vehicle of mass $m_1$ is moving on a horizontal plane, a stone of mass $m_2$ strikes the vehicle with a velocity of $v_2$ = 10 m/s, making an angle of 37° with the horizontal, as shown in the figure. What is the velocity of the vehicle and stone after the collision?



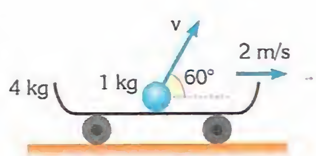

60. A car of mass 1000 kg initially at rest fires a ball of 100 kg with a velocity $v=50\sqrt{2}$ m/s at an angle of 45° to the horizontal, as shown in the figure. What will the velocity of the car be after firing a ball, relative to the ground?




61. While an 8 kg vehicle is moving together with a 1 kg object on a smooth horizontal plane, at a velocity of 12 m/s, as shown in the figure, the object is ejected at a velocity of v relative to the ground. If the velocity of the vehicle decreases to 10 m/s after the object is thrown, what is the velocity, v, of the object?




62. While a vehicle with a mass of 4 kg is moving at a velocity of 2 m/s, a 1 kg object is thrown out of the vehicle at a velocity of v relative to the ground, as shown in the figure.




If the velocity of the vehicle remains unchanged, what is the velocity, v? (Friction is neglected.)

63. A 48 kg child standing on a smooth surface throws a 1 kg ball at a velocity of 6 m/s to his friend at an angle of 37° to the horizontal.


a) What is the impulse acting on the child in the horizontal direction?

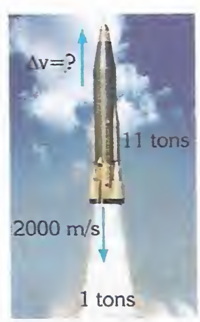
b) Neglecting the friction on the ground, at what velocity does the child start to move?

### 7.5 Rockets

64. What is the change in the velocity of a rocket, which ejects 1000 kg of gas at a velocity of 800 m/s, if its mass is 20000 kg after the ejection?

65. If there is a 10 m/s change in the velocity of a 30500 kg rocket, at what velocity is the 500 kg gas ejected?

66. What is the change in the velocity of a rocket of mass 11 tons, including the fuel, from which 1 ton of fuel is ejected at a velocity of 2000 m/s?



# Uniform Circular Motion and The Universal Law of Gravity

CHAPTER 8

*The Earth orbiting the sun and a spaceship or a satellite revolving around the Earth are well known examples of circular motion. Everyday devices also exhibit circular motion. Examples are the drum of a washing machine, a roundabout in an amusement park and a compact disc in an operating CD player. In this chapter circular motion of such objects will be examined. Additionally, the universal law of gravity, which causes objects to attract each other will be examined. Gravitational forces keep the celestial bodies in nearly circular motion.*

## 8.1 UNIFORM CIRCULAR MOTION

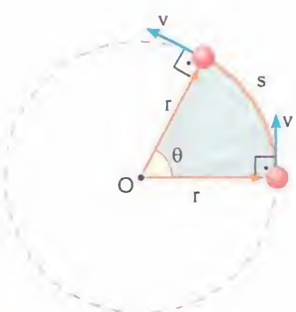



**Figure 8.1** As the radius vector undergoes circular motion it sweeps out an angle θ. The arc length, s, covered by the object during part of its circular path is $s = r\theta$

When an object revolves around a fixed axis at a fixed distance from it, r, as shown in Figure 8.1, it is described as **circular motion**. During this motion, as the object covers a distance s (called an arc) in the circular path, the radius vector r which joins the rotational axis and the object, sweeps out an angle θ. This angle is expressed as

$$\theta = \frac{s}{r}$$

it is measured in degrees or in radians. For one complete revolution of the object. The angle swept out by the radius vector is

a) in degree measurement: 360°.

b) in radian measurement: $2\pi$ radians.

The unit radian is less familiar but widely used in science.

From the equation $\theta = \frac{s}{r}$, one radian can be defined as follows

**One radian is the angle subtended by the arc whose length equals the radius of the circle.** See Figure 8.2.

$$\text{As} \quad 2\pi \ \text{radians} = 360° \quad \Rightarrow \quad 1\,\text{radian} = \frac{360°}{2\pi} = 57.3°$$

Since the angle in radians is defined as a ratio of two lengths: arc length to radius (from $\theta = \frac{s}{r}$), it has no dimension. The unit "radian" is carried only as a reminder that the angles are measured in radians and not in degrees.

The motion of an object moving in a circular path with a constant speed is defined to be **uniform circular motion.**

The time needed for one complete revolution of uniform circular motion is called the **period** and denoted as T. Its SI unit is the second.
The number of revolutions that an object makes per unit time is called the **frequency** f. The frequency is the inverse of the period T.

$$f = \frac{1}{T}$$

so the SI unit of frequency is $\frac{1}{s}$ = Hertz.

One hertz is defined as one cycle per second.

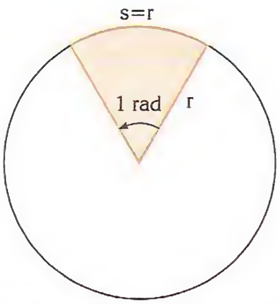


Figure 8.2 One radian is the angle subtended by the arc s whose length equals the radius of the circle.

## a. Tangential Speed and Angular Speed

Let's assume that the object exhibiting uniform circular motion in Figure 8.3 moves from point A to point B in a time interval t. The velocity vector of the object is always a tangent to the circular path, and is called **tangential velocity**. The magnitude of the tangential velocity, which is constant, is expressed as tangential speed and can be defined by the equation

$$v = \frac{\text{arc length}}{\text{time elapsed}} = \frac{s}{t}$$

In one complete revolution, $s = 2\pi r$ and $t = T$

Thus, the speed of the object in its circular path is

$$v = \frac{2\pi r}{T} \quad \text{or since} \quad f = \frac{1}{T} \quad \text{then} \quad v = 2\pi r f \quad \text{(tangential speed)}$$

The angle swept out by the radius vector per unit time is called the **angular speed** of the object. Hence, the angular speed is given as

$$\omega = \frac{\theta}{t},$$

In one complete revolution, $\theta = 2\pi$ and $t = T$, this can be re-writen as

$$\omega = \frac{2\pi}{T} = 2\pi f \quad \text{(angular speed)}$$

The relationship between the tangential and angular speeds can thus be found by substituting

$\omega$ for $\frac{2\pi}{T}$ into the equation $v = \frac{2\pi r}{T}$,

This leads to the following equation;

$$v = \omega r$$

where v is the tangential speed, $\omega$ is the angular speed and r is the radius of the circular motion.

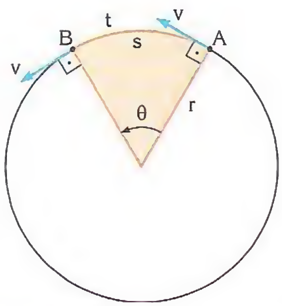


Figure 8.3 Circular motion has two speeds: tangential speed v and angular speed ω.

## Example 8.1 Frequency, period, angular and tangential speeds

The drum of a washing machine rotates 1200 times in 1 minute.

a) What is the period and frequency of the drum?

b) What is the angular speed of the drum?

c) If the diameter of the drum is 40 cm, what is the tangential speed of a point on the drum? (Take $\pi=3$)

### Solution

a) The drum rotates 1200 times in 1 minute. Since 1 minute is 60 s, the frequency of the drum is;

$$f = \frac{1200}{60\ s}, \text{ then } f = 20\ s^{-1} = 20\ Hz$$

Hence, the period of the drum is;

$$T = \frac{1}{f} = \frac{1}{20\ Hz}, \text{ then } T = 0.05\ s$$

b) The angular speed of the drum is;

$\omega=2\pi f =2(3\ rad/s)(20\ Hz)$ then $\omega=120\ rad/s$.

c) Since the diameter of the drum is 40 cm, the distance of any point on the drum to the axis of rotation is 20 cm. Thus, the radius of the drum is $r=20\ cm=0.2\ m$.

Therefore, the tangential speed of a point on the drum is;

$v = \omega \cdot r = (120\ rad/s)(0.2\ m)$ thus $v = 24\ m/s$

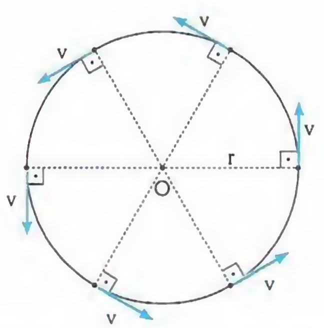



Figure 8.4 The continuously changing direction of the velocity vectors in uniform circular motion cause a change in velocity, hence an acceleration.

## b. Centripetal Acceleration

An object performs uniform circular motion, as shown in Figure 8.4, although the magnitude of its velocity (that is, its speed) remains constant, the direction of the velocity is continuously changing.

Chapter 2 explained that acceleration is the change in velocity per unit time. A change in either the magnitude or the direction of velocity produces an acceleration. Thus, an object experiencing uniform circular motion is continuously accelerating. This acceleration will now be examined quantitatively.

Let us assume that an object exhibiting uniform circular motion moves from point A to point B in a time interval $\Delta t$, as shown in Figure 8.5.a. The instantaneous

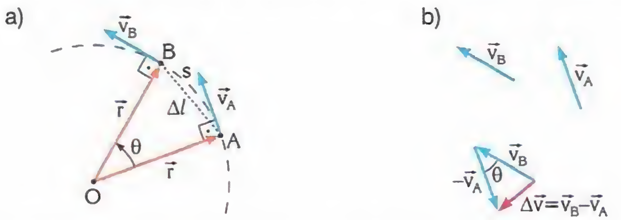



**Figure 8.5** Determining the change in the velocity, $\Delta\vec{v}$, for an object moving in a circle. a) The instantaneous velocity vectors $\vec{v}_A$ and $\vec{v}_B$ have the same magnitude but differ in direction. b) For small angles $\theta$, $\Delta\vec{v}$ is perpendicular to both $\vec{v}_A$ and $\vec{v}_B$

velocity vectors at points A and B, $\vec{v}_A$ and $\vec{v}_B$, have the same magnitude but differ in direction. During time interval $\Delta t$, the change in velocity is

$$\Delta\vec{v}=\vec{v}_B-\vec{v}_A$$

In order to apply the vector addition, as shown in Figure 8.5.b, this equation can be re-written as

$$\Delta\vec{v}=\vec{v}_B+(-\vec{v}_A)$$

Since acceleration is

$$\vec{a}=\frac{\Delta\vec{v}}{\Delta t}$$

If the time interval $\Delta t$ is decreased, points A and B come closer, reducing the angle $\theta$. From Figure 8.5.b it can be seen that this angle is also the angle between the two velocity vectors, since radius and velocity vectors are always perpendicular. Eventually angle $\theta$ becomes so small that $\vec{v}_A$ and $\vec{v}_B$ are almost parallel and their difference $\Delta\vec{v}$ is almost perpendicular to both of them. Hence, instantaneous acceleration, which is in the same direction as $\Delta\vec{v}$, is directed towards the centre of the circular path. Therefore a particle moving in a circular path is always accelerated toward the centre and the acceleration is always perpendicular to the direction of the velocity. This acceleration is called the **centripetal acceleration** (centre-seeking acceleration).

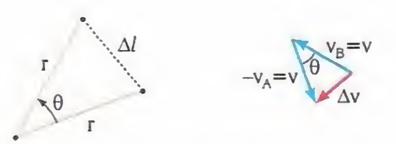



**Figure 8.6** Since the angles swept out by both the radius vector and the velocity vector are the same, the triangles formed by these vectors are similar.

From the similarities of the two triangles in Figure 8.6

$$\frac{\Delta l}{r}=\frac{\Delta v}{v}$$

Since $\theta$ is very small, $\Delta l \cong s$ and also $\Delta l = s = v\cdot\Delta t$

Thus $v\Delta t$ can be substituted for $\Delta l$ in the equation above.

$$\frac{v\Delta t}{r}=\frac{\Delta v}{v}$$

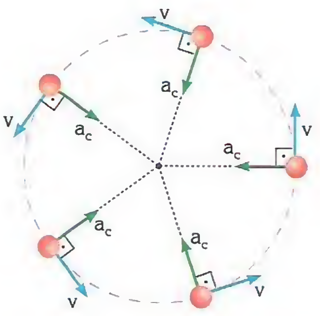



**Figure 8.7** An object undergoing circular motion experiences a centripetal acceleration.

after rearranging this equation

$$\frac{\Delta v}{\Delta t} = \frac{v^2}{r} \quad \text{where} \quad \frac{\Delta v}{\Delta t} = a, \text{ so} \qquad a_c = \frac{v^2}{r} \quad \text{(centripetal acceleration)}$$

where the subscript c denotes centripetal acceleration. As stated above, this acceleration is always directed towards the centre of the circular path, and is always perpendicular to the velocity, as shown in Figure 8.7.

Since $v = \omega r$, $a_c$ can also be expressed as,

$$a_c = \frac{v^2}{r} = \omega^2 r$$

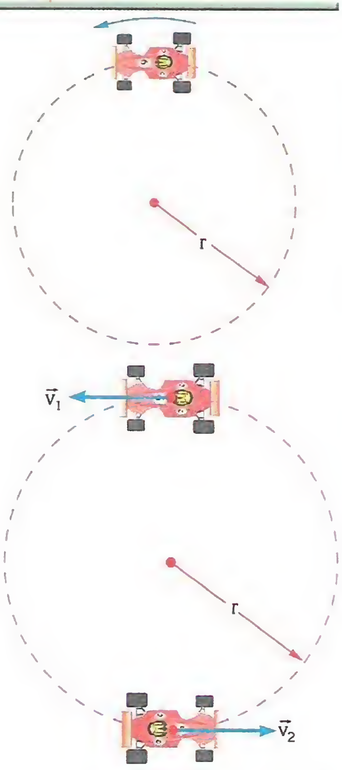
## Example 8.2 Average and centripetal accelerations

A toy car is following a circular path with a constant speed of v=3 m/s and a period of 2 s, as shown in the figure. If the radius of the circular path is 1 m, find

a) The centripetal acceleration of the car,

b) The change in tangential velocity $\Delta\vec{v}$ in 1 s

c) The average acceleration of the car in 1 s.




**Solution**

a) Applying the centripetal acceleration equation for circular motion;

$$a = \frac{v^2}{r} = \frac{(3\text{ m/s})^2}{1\text{ m}}$$
$$a = 9\text{ m/s}^2$$

The direction of this acceleration is towards the centre of the circular path.

b) Since the period of the motion is 2 s, the car will cover a semicircle in 1 s, and the directions of the initial and the final velocity vectors will be opposite each other. Let the initial and final velocities be $\vec{v}_1$ and $\vec{v}_2$, as shown on the right-hand diagram. However, since both velocity vectors have equal magnitudes, they can both be represented as $\vec{v}$. Therefore the change in tangential velocity is

$$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1 = \vec{v} - (-\vec{v}) = 2\vec{v}$$

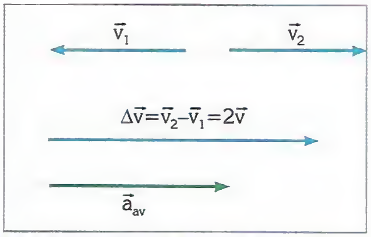
Its magnitude is

$$\Delta v = 2v = 2\,(3\text{ m/s})$$
$$\Delta v = 6\text{ m/s}$$

c) The average acceleration in 1s is

$$a_{av} = \frac{\Delta v}{\Delta t} = \frac{6\text{ m/s}}{1\text{ s}} \quad \text{thus} \quad a_{av} = 6\text{ m/s}^2$$

The direction of this acceleration is the same as that of vector $\Delta\vec{v}$. Their respective diretions are shown on the right-hand diagram.



## c. Centripetal Force

Force is required to produce centripetal acceleration.

According to the second law of motion ($\vec{F}_{net} = m\vec{a}$), an accelerating object must have a net force acting on it. Since an object exhibiting uniform circular motion always accelerates towards the centre of a circle, it must always experience a net force, as shown in Figure 8.8. This force keeps the object in a circular path and is called the **centripetal force**, $\vec{F}_c$. Its magnitude can be obtained by applying the second law of motion. Thus,

$\vec{F}_{net} = m\vec{a}$ where a is centripetal acceleration,

$$a_c = \frac{v^2}{r} \text{ or } a_c = \omega^2 r \qquad F_c = \frac{mv^2}{r} \text{ or } F_c = m\omega^2 r$$

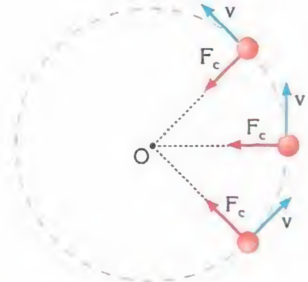



Figure 8.8 The centripetal force keeps an object in uniform circular motion and always points toward the centre of a circular path, in the same direction as centripetal acceleration.

This force is obviously necessary because otherwise, if no force acted on the object, it would not move in a circle but in a straight line as the first law of motion describes. Centripetal force is not a new type of force, but rather one of the many forces that have already been discussed. However, it is called 'centripetal' because of its direction, which is always towards the centre of a circular path.

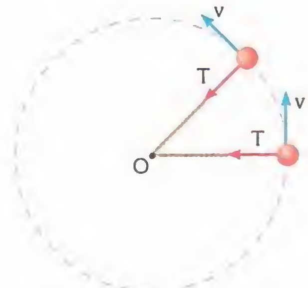



Figure 8.9 The tension in the string is the force that gives rise to the centripetal acceleration. Therefore, the tension is responsible for the centripetal force here.

A good example of the action of a centripetal force is the tension force in a string that keeps an object in circular motion on a horizontal frictionless surface, as shown in Figure 8.9. The tension in this case is responsible for the centripetal force. Another good example is the gravitational force exerted on the Moon by the Earth. This acts as the centripetal force that keeps the Moon in circular motion around the Earth.

There is a common misconception that an object moving in a circle experiences an outward force called the **centrifugal force** (force which acts away from the centre). Consider the example below:

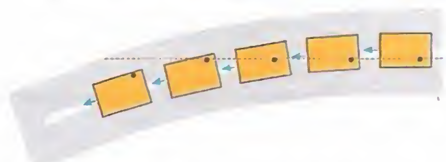
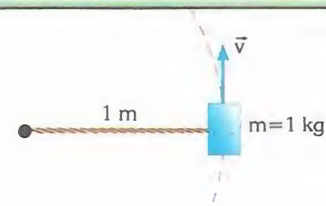


Figure 8.10 Centrifugal force actually occurs as a result of inertia. A person in a car, which is originally travelling on a straight road, tends to remain on the straight path while the car is making a turn and the person feels' as if an outward acting force is exerted upon him.

Imagine that you are a passenger sitting in the back seat of a car which is making a left-hand turn, as shown in Figure 8.10. Also assume that there is no friction between you and the seat. As the car begins to take the turn to the left, you begin sliding to the right, it seems to you that you are being forced to the right. In fact, as the car turns to the left, your inertia keeps you in a straight line path (red dashed line drawn in the figure). In this case, you will slide across the seat until you reach the right-hand passenger door to try to maintain the straight line path on the red dashed line. Once you reach the right-hand door, a force is applied upon you by the door to set you in a circular path. This force exerted by the door is the centripetal force. Actually, no right-hand or outward force (so-called centrifugal force) acts on you.

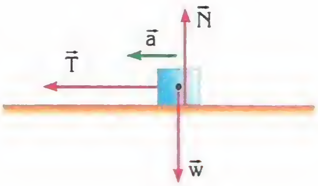
### Example 8.3 — Uniform circular motion in a horizontal plane

A 1 kg block is attached to the end of a 1 m long rope and is moving in a circular path on a smooth horizontal plane with an angular speed of $\omega = 2\pi$ rad/s, as shown in the figure. Draw the forces acting on the system and calculate the tension in the rope. (Take $\pi = 3$)



### Solution

The net force applied to the block on the smooth horizontal plane is the tension in the rope. It can be seen in the force diagram, that both gravitational force $\vec{w}$ and the normal contact force vector are acting upon it. However, the sum of these forces is zero. The acceleration of this block is in the same direction as the tension, which acts towards the centre. Thus, the magnitude of this acceleration is;



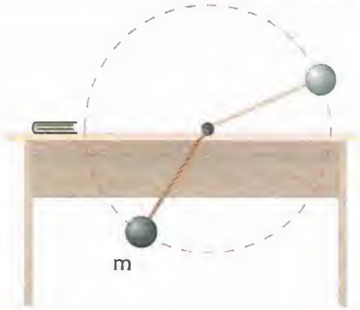

$$a = \omega^2 r = (2\times3 \text{ rad/s})^2(1 \text{ m}) = 4\pi^2 = 36 \text{ m/s}^2$$

Therefore, the magnitude of the tension in the rope is;

$$T = ma = (1 \text{ kg})(36 \text{ N/kg}) \quad \text{thus} \quad T = 36 \text{ N}$$

## Example 8.4 — Uniform circular motion in a vertical plane

An object of mass 1 kg rotating in a vertical plane with a constant speed of 10 m/s at the end of a rope is shown in the figure. If the radius of the circular motion is 0.5 m, calculate the tension in the rope at the highest and the lowest points of the trajectory.




### Solution

At the top, all forces are directed toward the centre

$$\Sigma\vec{F}_y = m\vec{a}_y \quad \text{where} \quad a_y = a = \frac{v^2}{r} \qquad w + T = ma$$

$$T = m\frac{v^2}{r} - mg \qquad \text{(minimum tension)}$$

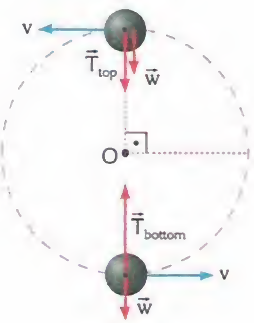
$$T = (1 \text{ kg})\frac{(10 \text{ m/s})^2}{0.5 \text{ m}} - (1 \text{ kg})(10 \text{ N/kg}) = 10 \text{ N}$$

at the bottom

$$\Sigma\vec{F}_y = m\vec{a}_y \quad \text{where} \quad a_y = a \qquad T - w = m\frac{v^2}{r}$$

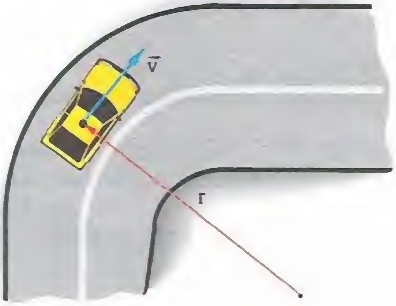
$$T = m\frac{v^2}{r} + mg \qquad \text{(maximum tension)}$$

$$T = (1 \text{ kg})\frac{(10 \text{ m/s})^2}{0.5 \text{ m}} + (1 \text{ kg})(10 \text{ N/kg}) = 30 \text{ N}$$




## Example 8.5 — A car driving round a curved road

For a car to drive round a curved flat road with a radius of 50 m at a speed of 20 m/s safely, what must the coefficient of static friction be?



Only the force of static friction provides the centripetal force, as shown in the figure.


(This force of friction is the force of static friction because the car does not move along the direction of the friction force while driving round the curve.)

The force of static friction must be sufficient to give the car an acceleration, thus,

$$a = \frac{v^2}{r} = \frac{(20\ m/s)^2}{50\ m} = 8\ m/s^2$$

We know that the friction force between the car and the road is;

$$f_s = mg\mu_s$$

$$\vec{F}_{net} = m\vec{a} \quad \text{where} \quad a = \frac{v^2}{r}$$

$$f_s = m\frac{v^2}{r}$$

$$mg\mu_s = ma$$

substituting the values in the equation, the coefficient of static friction is obtained

$$\mu_s \cdot (10\ m/s^2) = 8\ m/s^2$$

$$\mu_s = 0.8$$

## 8.2 THE UNIVERSAL LAW OF GRAVITY

Why do objects fall towards Earth when released? There are thousands of artificial satellites revolving around the Earth. What keeps a satellite in its orbit? Why are the celestial bodies such as the Sun, the planets and the Moon spherical in shape? The law of gravitation provides the answers to these questions and many others.

### a. The Universal Law of Gravity

The most familiar example of gravitational attraction is an object falling towards Earth when released from a height. In the seventeenth century, Isaac Newton discovered that the same type of force that makes an apple fall out of a tree holds the planets in their orbits around the Sun. However, in this age many scientists had trouble accepting the idea of a force acting at a distance. They knew that forces could act through contact such as the force of hands acting on a table to pull it in a given direction, or a foot kicking an object, and so on. Newton's ground breaking idea was that gravity acts without any form of contact. Newton proposed that the Earth exerted a force upon all objects surrounding us, such as a falling apple and even the Moon, though there is no physical contact. Newton then discovered the fundamental character of the force of gravitational attraction between two objects. In 1686 he stated the law of gravitation as follows:

**Every particle in the universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. This force acts along the line joining the centres of the particles.**

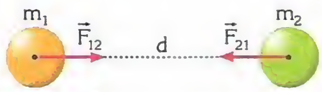



Figure 8.11 Two particles, separated by a distance d, exert a gravitational attractive force upon each other. The forces on each particle are equal in magnitude but opposite in direction, $\vec{F}_{12} = -\vec{F}_{21}$.

If the particles have masses $m_1$ and $m_2$ and are separated by a distance d, as shown in Figure 8.11, the magnitude of the gravitational force between them is

$$F = G\frac{m_1 m_2}{d^2} \quad \text{the universal law of gravitation}$$

Here, G is a universal constant called the universal gravitational constant. This has been measured experimentally as

$$G = 6.67\times10^{-11}\ Nm^2/kg^2$$

Large celestial bodies such as moons, planets and stars, tend to be spherical since all particles in a large celestial body gravitationally attract each other. This causes the minimization of the distance between them. As a result, the body naturally assumes a spherical shape, just as a handful of snow forms a spherical shape if it is squeezed with equal forces on all sides. Much smaller size celestial bodies, such as meteors, are usually not spherical since the gravitational attraction between the particles is too small.

## b. The Gravitational Field

There are two ways of finding the gravitational force between two particles, as shown in Figure 8.11.

1. Direct interaction: The law of gravity can be directly stated

$$F = G\frac{m_1 m_2}{d^2}$$

2. Field interaction: In order to find the gravitational force between two objects, the gravitational field must first be defined. One of the particles, lets say particle 1 of mass $m_1$, sets up a gravitational field covering every point in space around it. This field is represented by lines which are directed toward the centre of mass $m_1$. When the other particle of mass $m_2$ is placed at a point in space around $m_1$, $m_2$ experiences a gravitational force $F=m_2g$, as shown in Figure 8.12. In other words, the gravitational field set up by $m_1$ exerts a force on $m_2$. Thus, using the fact that $F=m_2g$ the gravitational field of $m_1$ can be obtained as

$$g = \frac{F}{m_2} \quad \text{Gravitational field strength of } m_1$$

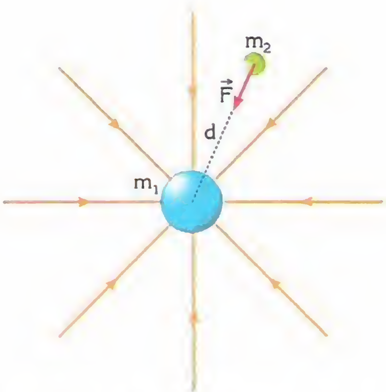



Figure 8.12 The gravitational field is a region around a massive object of mass $m_1$ in which another object of mass $m_2$ would feel a gravitational force of attraction. Gravitational field lines around an object are directed towards the centre of the object.

Assuming that $m_2$ is very small with respect to $m_1$, that is, if $m_2$ is a unit mass, the gravitational field of $m_1$ can be defined as follows;

**the gravitational field at a point in space is the gravitational force per unit mass at that point.**

For example, when an object of mass m is placed near the surface of the Earth, it experiences a gravitational force F=mg directed toward the centre of the Earth. The gravitational force, from the law of gravitation, between the Earth and the object is

$$F = \frac{GM_E m}{R_E^2} \quad \text{where } M_e \text{ is the mass of the Earth and } R_E \text{ is the radius of the Earth}$$

Thus, the gravitational field of the Earth near the surface is

$$g = \frac{F}{m} = \frac{GM_E}{R_E^2}$$

After finding the gravitational field g at a location, which is also called the gravitational acceleration of the Earth, the gravitational force that an object of mass m experiences at that location can be found as

$$\vec{F} = m\vec{g}$$

This force is directed toward the centre of the Earth. It is also called the weight of the object.

The gravitational field in the space around a uniform spherical mass, such as a planet, varies in both direction and magnitude, as shown in Figure 8.13.a. However, at a location near the surface of the planet the magnitude of the gravitational field is assumed to be constant and the gravitational field vectors are parallel, as shown in Figure 8.13.b.

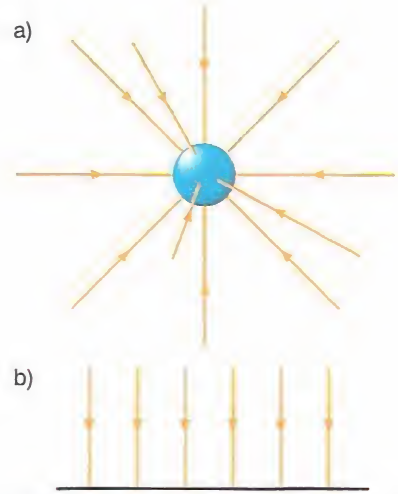
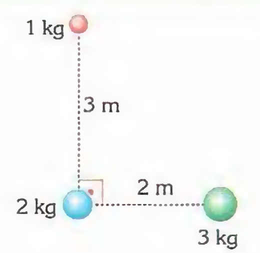



**Figure 8.13.** a) The gravitational field around a uniform spherical mass varies in both direction and magnitude. b) At a location near the surface of a massive mass, such as a planet, the magnitude of the gravitational field is assumed to be constant and the gravitational field lines are parallel.

## Example 8.6 — Gravitational force

According to the data given in the right-hand figure, find the magnitude and direction of the net force acting on the 2 kg mass. ($G = 6.67\times10^{-11}$ Nm²/kg²)




### Solution

First calculate the forces acting on the 2 kg mass separately. Their directions will be as shown in the figure. And their magnitudes are

$$F_1 = G\frac{m_1 m_2}{d^2}$$

$$= (6.67\times10^{-11}\ \mathrm{Nm^2/kg^2})\frac{(1\ \mathrm{kg})(2\ \mathrm{kg})}{(3\ \mathrm{m})^2}$$

$$= 1.5\times10^{-11}\ \mathrm{N}$$

$$F_2 = (6.67\times10^{-11}\ \mathrm{Nm^2/kg^2})\frac{(3\ \mathrm{kg})(2\ \mathrm{kg})}{(2\ \mathrm{m})^2}$$

$$= 10\times10^{-11}\ \mathrm{N}$$

Hence, the direction of the resultant force will be as shown in the figure and its magnitude is obtained from pythagorean theorem;

$$F_{net}^2 = F_1^2 + F_2^2$$

$$F_{net}^2 = (1.5^2 + 10^2)(10^{-11})^2$$

$$F_{net} \cong 10^{-11}\ \mathrm{N}$$

Thus, the gravitational attraction is not significant for small masses and short distances.

# Example 8.7 

Calculate the mass of the Earth by applying the law of gravitation and gravitational acceleration. (Take $g=10\ m/s^2$, $G=6.67\times10^{-11}\ Nm^2/kg^2$ and $R_E=6.4\times10^3$ km)


### Solution

The weight of an object on the surface of Earth is calculated from w=mg.
and the gravitational force between the object and Earth is obtained from the universal law of gravitation.

$$F = G\frac{mM_E}{R_E^2}$$

Equating the weight and the gravitational force, the mass of Earth can be found.

$$mg = G\frac{mM_E}{R_E^2} \Rightarrow M_E = \frac{gR_E^2}{G} = \frac{(10\ m/s^2)(6.4\times10^6\ m)^2}{6.67\times10^{-11}\ Nm^2/kg^2}$$

Thus, the mass of Earth is $M_E = 6\times10^{24}$ kg

## c. Satellites

A satellite is an object orbiting the Earth, the Sun or another massive body. Satellites can be classified as natural or artificial satellites. Examples of natural satellites are the Moon orbiting the Earth, planets orbiting the Sun. Artificial satellites are constructed by engineers and launched from the Earth for various purposes such as communication, scientific research, weather forecasting, gathering intelligence, etc. Although each satellite has a different function, they all operate on the same physics principles and are described by the same mathematical equations.

In order to understand how a satellite follows its orbit, consider a projectile launching horizontally from the top of a mountain, neglecting air friction, as shown in Figure 8.14. The range of the projectile depends upon the launch speed: the greater the launch speed, the greater the range. Figure 8.14 shows various trajectories for different launch speeds. As the launch speed is increased, a speed is reached which sets the projectile in a circular path. At this speed, the motion of the projectile can be regarded as falling towards Earth at the same rate as Earth curves away from it. This causes the projectile to stay at the same height above Earth and to orbit in a circular path. A satellite is also a projectile, upon which the

**Figure 8.14** The trajectories of a projectile launched from the top of a mountain with various launch speeds. At a given speed, the projectile can be set in a circular path

only force acting is gravity. This is then the centripetal force, as shown in Figure 8.15.

What launch speed does a satellite require to orbit the Earth? The radius of the orbit is r, measured from the centre of the Earth, the centripetal acceleration of the satellite is $a = \frac{v^2}{r}$. From the law of gravity, the gravitational force between the Earth and the satellite of mass m is

$$F_g = \frac{GM_E m}{r^2}$$

This force is always directed towards the centre of the Earth. Using the second law of motion

$$\vec{F}_{net} = m\vec{a} \Rightarrow \frac{GM_E m}{r^2} = \frac{mv^2}{r}$$

Solving this equation for v, we obtain

$$v = \sqrt{\frac{GM_E}{r}}$$

This equation indicates that the orbit radius r and the linear speed v are dependent upon each other. For a given radius r, the speed v of a circular orbit must be determined from the equation above.

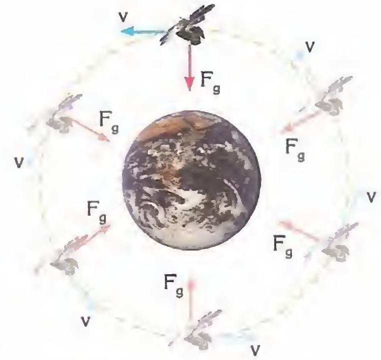



**Figure 8.15** The gravitational force between the satellite and the Earth gives rise to the centripetal force and keeps the satellite in a circular orbit

## Example 8.8 — Geosynchronous satellite

A geosynchronous satellite is one that stays above the same point of the Earth. If the mass of the Earth is $M_E = 5.98\times10^{24}$ kg, and the universal gravitational constant is $G = 6.67\times10^{-11}$ Nm²/kg², determine

a) The height above the Earth's surface that a geosynchronous satellite must orbit

b) The linear speed of the satellite.

**Solution**

a) The only force on the satellite is the gravitational force. Assuming that the satellite moves in a circle, the second law of motion can be applied

$$\vec{F}_{net} = m\vec{a}$$

$$G\frac{M_E m_s}{r^2} = m_s\frac{v^2}{r}$$

Since the period of the satellite is

$$T = 1 \text{ day} = (24 \text{ h})(3600 \text{ s/min}) = 86400 \text{ s},$$

the speed of the satellite can be found from the equation

$$v = \frac{2\pi r}{T}$$

Substituting this into the first equation above and solving for r

$$G\frac{M_E}{r^2} = \frac{(2\pi r)^2}{rT^2}$$

$$r^3 = \frac{GM_E T^2}{4\pi^2}$$

$$= \frac{(6.67\times10^{-11}\ \text{Nm}^2/\text{kg}^2)(5.98\times10^{24}\ \text{kg})(86400\ \text{s})^2}{4\pi^2}$$

$$= 7.54\times10^{22}\ \text{m}^3$$

Taking the cubed root of both sides of this equation

$$r = 4.23\times10^7 \text{ m} = 42\,300 \text{ km}$$

from the Earth's centre. That is, about 36 000 m from the Earth's surface.

b) From the equation

$v = 2\pi r/T$ where

$r = 4.23\times10^7$ m and $T = 86\,400$ s,

$v = 3\,070$ m/s

## 8.3 GENERAL FORM OF GRAVITATIONAL POTENTIAL ENERGY

In Chapter 6.4 the concept of gravitational potential energy was discussed and the equation for the potential energy of an object was found to be

$$PE = mgh$$

where m is the mass of the object, g is the gravitational acceleration and h is the height of the object above or below some reference level.

However, this equation is valid only when the object is near the surface of Earth.

For objects far away from the surface of Earth, such as a satellites, another expression must be used to obtain the gravitational energy. The general equation for the gravitational potential energy of an object of mass m a distance r from the centre of the Earth can be expressed as

$$PE = -\frac{GM_E m}{r}$$

where $M_E$ is the mass of the Earth.

The reference level for this equation is that potential energy is zero at an infinite distance from the centre of the Earth. This means that, at a very large distance from Earth the potential energy is zero (there is no gravitational attraction).

## 8.4 BINDING ENERGY

Let's consider the circular motion of a body of mass m about a much larger body M, that is, $M \gg m$. For example, the motion of the Moon or an artificial satellite around the Earth or the motion of a planet around the Sun, assuming the massive body to be stationary. It is also assumed that the paths of the satellites are nearly circular.

The potential energy of the system is

$PE = -\frac{GMm}{r}$ where r is the radius of the circular path

The kinetic energy of the satellite moving with linear speed v is

$$KE = \frac{1}{2}mv^2$$

And the total mechanical energy of the system is

$$E = KE + PE = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Only gravitational force, $\vec{F}_g$, provides the centripetal acceleration, $a_c = \frac{v^2}{r}$, of the revolving body so

$$\vec{F}_{net} = m\vec{a} \Rightarrow \vec{F}_g = m\vec{a}_c \Rightarrow \frac{GMm}{r^2} = \frac{mv^2}{r}$$

The last equation can be rearranged to obtain an expression for KE as

$$\frac{mv^2}{2} = \frac{GMm}{2r} \quad \text{(thus, kinetic energy is half of the potential energy)}$$

Substituting $\frac{GMm}{2r}$ for $\frac{mv^2}{2}$ into the equation for total mechanical energy

$$E = KE + PE = \frac{GMm}{2r} - \frac{GMm}{r}$$

Total mechanical energy is

$$E = -\frac{GMm}{2r}$$

This energy is constant and negative. Potential energy is always negative, except at infinite separation where it is zero. Kinetic energy can never be negative. It approaches zero as the separation approaches infinity.

The absolute value of E is also equal to the binding energy of the system

$$E_{binding} = \frac{GMm}{2r}$$

That is, as a body of mass m revolves around a much larger body of mass M, it becomes bound to the larger mass, M with an energy that is given by the above equation.

Alternatively, the binding energy is the minimum energy required by a bound system in order to separate its components and send them to infinity.

## 8.5 THE ESCAPE SPEED

When an object is thrown vertically upwards, its velocity decreases to zero at a maximum height, and then it falls back downwards. If the object is thrown at greater and greater speeds, there will be a speed called the **escape speed** at which the object will not return. The speed of the object must be such that its maximum height is at infinity, where its speed decreases to zero. Here infinity refers to a very far point where the gravitational field of the Earth is accepted to be zero.

Neglecting air resistance, the conservation of mechanical energy can be applied, since the only force acting on the object is the gravitational force which is conservative

$$E_i = E_f$$

$$KE_i + PE_i = KE_f + PE_f$$

$$\frac{1}{2}mv_i^2 - \frac{GM_Em}{r_i} = \frac{1}{2}mv_f^2 - \frac{GM_Em}{r_f}$$

When thrown at the minimum speed $v_i$ (also called the escape speed), the object must reach infinity with a final speed of zero.

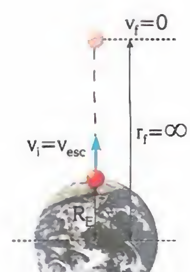



**Figure 8.16** For an object to escape from the influence of the Earth's gravitational field, it must be thrown vertically upwards with a minimum initial speed, called the escape speed. Its speed decreases to zero at infinity

That is, $v_f = 0$ and $r_f = \infty$ for the minimum initial speed, and $r_i = R_e$, $v_i = v_{esc}$ the result is

$$\frac{1}{2}mv_{esc}^2 - \frac{GM_Em}{R_E} = 0 - 0$$

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}} \quad \text{Escape speed}$$

When this speed is attained by an object, whatever its mass is, it can escape from the influence of the Earth's gravitational field.

## Example 8.9 — Gravitational potential energy

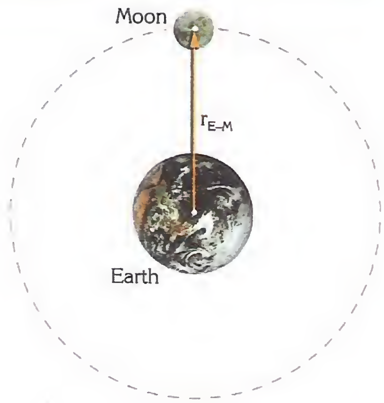
Using the information concerning the Earth and the Moon given below, calculate

a) The gravitational potential energy between the Earth and the Moon,

b) The binding energy of the Moon to the Earth,

c) The speed required by the Moon to liberate it from Earth.

($M_E = 6\times10^{24}$ kg, $m_M = 7.5\times10^{22}$ kg, $r_{E-M} = 3.8\times10^8$ m, $G = 6.67\times10^{-11}$ Nm²/kg²)




### Solution

a) The gravitational potential energy between the Earth and the Moon is

$$PE = -G\frac{M_E m_M}{r_{E-M}}$$

$$PE = -(6.67\times10^{-11}\ Nm^2/kg^2)\frac{(6\times10^{24}\ kg)(7.5\times10^{22}\ kg)}{(3.8\times10^8\ m)}$$

$$PE \cong -8\times10^{28}\ J$$

b) The total energy of the Moon on orbiting the Earth is the sum of its kinetic and potential energies. Binding energy is the negative of this total energy. Thus

$$E_{bin} = G\frac{M_E m_M}{2r_{E-M}}$$

$$E_{bin} = (6.67\times10^{-11}\ Nm^2/kg^2)\frac{(6\times10^{24}\ kg)(7.5\times10^{22}\ kg)}{2(3.8\times10^8\ m)}$$

$$E_{bin} \cong 4\times10^{28}\ J$$

c) The equation for the escape speed is

$$v_{esc} = \sqrt{\frac{GM_E}{r_{E-M}}} = \sqrt{\frac{(6.67\times10^{-11}\ Nm^2/kg^2)(6\times10^{24}\ kg)}{3.8\times10^8\ m}}$$

$$v_{esc} = 1000\ m/s$$

# Summary

The motion of an object moving in a circular path at a constant speed is defined to be **uniform circular motion**.

The time required for one complete revolution is called the **period** and is denoted as T. Its unit is the second. The number of revolutions the object makes in unit time is called the **frequency** f. The frequency is the inverse of the period T.

$$T = \frac{1}{f}$$

The SI unit of frequency is the Hertz.

The speed of an object in circular motion is called the tangential speed. It is expressed as

$$v = \frac{2\pi r}{T}$$

The angle swept out by the radius vector per unit time is called the **angular speed** of the object. Hence, the angular speed is given by

$$\omega = \frac{2\pi}{T}$$

The relationship between tangential and angular speed is

$$v = \omega r$$

An object rotating about a point with a constant speed v experiences a centripetal acceleration

$$a = \frac{v^2}{r}$$

where r is the radius of circular motion.

Centripetal acceleration is caused by centripetal force

$$F_c = \frac{mv^2}{r}$$

The universal law of gravitation discovered by Newton states that

**Every particle in the universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. This force acts along the line joining the particles.**

$$F = \frac{Gm_1m_2}{r^2}$$ **the universal law of gravitation**

The expression for the gravitational potential energy of an object of mass m at a distance r from the centre of the Earth is

$$PE = -\frac{GM_Em}{r}$$

where $M_E$ is the mass of the Earth.

The binding energy for circular motion of a body of mass m about a much larger body of mass M is

$$E_{binding} = \frac{GMm}{2r}$$

The minimum speed that must be given to an object to liberate it from the influence of the gravitational field of the Earth is called the escape speed.

$$v = \sqrt{\frac{GM_E}{R_E}}$$

# QUESTIONS AND PROBLEMS

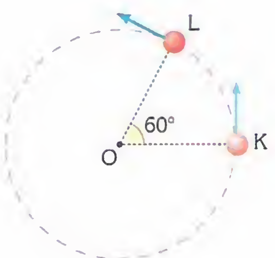
8.1 Uniform Circular Motion

1. What is uniform circular motion?

2. What is one radian in terms of degrees?

3. On a merry-go-round, is a child near the centre of the machine or near the edge of the machine travelling faster?

4. An object exhibiting uniform circular motion, as shown in the figure, travels from point K to point L in 1s. Find the following parameters of circular motion




a) Period
b) Frequency
c) angular speed.
d) What is the tangential speed of the object if the radius of the circular motion is 0.5m?

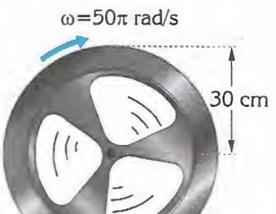
(Take $\pi$ to be 3)

5. An engine rotates 3600 times per minute. Calculate the
a) period in seconds
b) frequency in Hertz
c) angular speed

(Take $\pi$ to be 3)

6. Calculate the periods and the frequencies of all hands (the hour hand, the minute hand and the second hand) of a table clock?

7. An iron wheel of radius 30 cm is rotating about an axis at an angular speed of $50\pi$ rad/s, as shown in the figure, find




a) The frequency and the period of the wheel.
b) The tangential speed of a point 10 cm away from the axis of rotation
c) The tangential speed at the rim of the wheel.

(Take $\pi$ to be 3)

**8.**

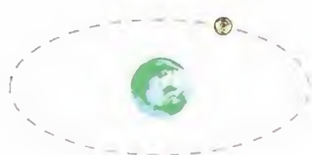
If the frequency of the rotor of a helicopter is 1200 cycles/min,

a) calculate the period of its rotor in seconds

b) if the rotor arm is 2.5 m long, find the tangential speed of a point at the far end of the arm.

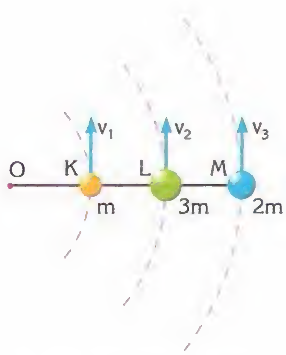
(Take $\pi$ to be 3)

**9.**



The period of the Moon orbiting the Earth is 27.3 days. If the Moon is 384 000 km away from the Earth, calculate its tangential speed. (Take $\pi$ to be 3)

**10.**

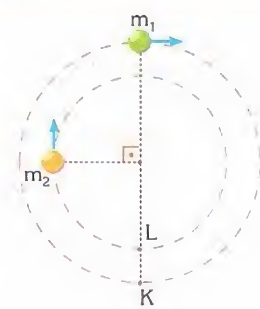



Three objects of different mass attached to the same string are revolving around point O, as shown in the figure. If |OK|=|KL|=|LM|, find the relationship between

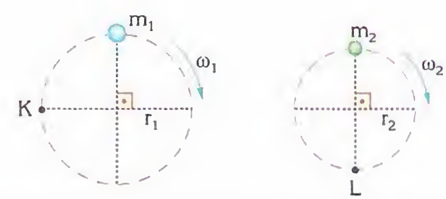
a) their angular speeds, $\omega_1$, $\omega_2$, and $\omega_3$.

b) their tangential speeds, $v_1$, $v_2$, $v_3$.

**11.** Masses $m_1$ and $m_2$ are rotated in a horizontal plane. After passing through the positions, shown in the figure, at the same instant when mass $m_1$ reaches point K, mass $m_2$ reaches point L. If the period of mass $m_1$ is 18 s, what is the period of mass $m_2$?

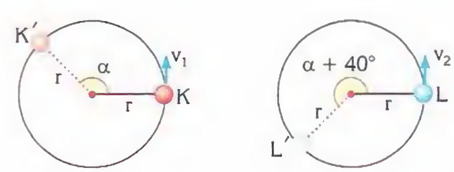



**12.**




The objects of masses $m_1$ and $m_2$ start performing uniform circular motion simultaneously with periods of $T_1$ and $T_2$ from the positions shown in the figure. At the moment the object of mass $m_1$ reaches point K, the object of mass $m_2$ reaches point L. What is the ratio of the angular speeds, $\omega_1/\omega_2$?

**13.**




Masses K and L, which are exhibiting uniform circular motion in a horizontal plane with periods $T_K$ and $T_L$, pass through points K and L at the same time. At a time 2 seconds later the masses take the positions K' and L', as shown in the figure.

If the ratio of their periods is $\frac{T_K}{T_L} = \frac{4}{3}$, what is the angle $\alpha$?

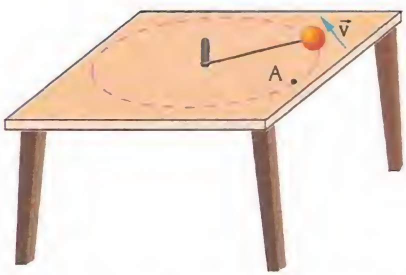
**14.** If a body attached to a string, and undergoing circular motion, covers equal lengths of arc within equal time intervals in a circular path. Which of the following parameters of the motion are constant?

a) speed

b) velocity

c) acceleration. What causes this acceleration of the body, and thus keeps it in a circular path?

**15.**




An object starting to move from point A on a frictionless table, is rotating in a circular orbit with a constant speed of 3m/s. Find the change in the tangential velocity when the object rotates by

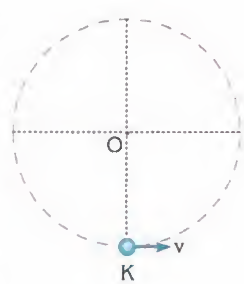
a) $\pi/3$,

b) $\pi/2$,

c) $\pi$,

d) $2\pi$.

**16.** The period of a 2 kg object exhibiting uniform circular motion at a speed of v=3 m/s is 12 s on a horizontal frictionless surface. If this object is at point K at t=0 as shown in the figure,




a) what is the change in its velocity between t=0 and t=34 s?

b) find its average acceleration between t=0 and t=34 s?

c) find the centripetal acceleration it experiences.

d) why are the average acceleration and centripetal acceleration different?

e) find the centripetal force.

(Take $\pi$ to be 3)

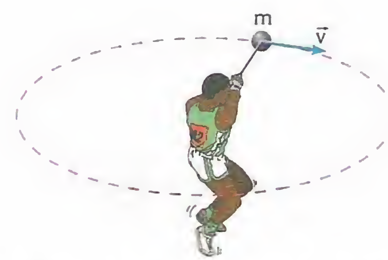
**17.** An object of mass m=2 kg rotates with a constant period of T=4 s on a horizontal frictionless surface, as shown in the figure. If the radius of the circular trajectory is 2 m, find


a) the centripetal acceleration

b) the centripetal force which the object experiences.

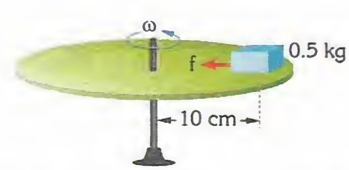
(Take $\pi$ to be 3)

**18.**




While an athlete is rotating a 0.5 kg stone at the end of a 1 m long rope at an increasing speed, as shown in the figure, the rope snaps at the tangential speed of 12 m/s. If the stone is rotated in a horizontal plane, find the maximum tension that the rope can bear.

**19.**

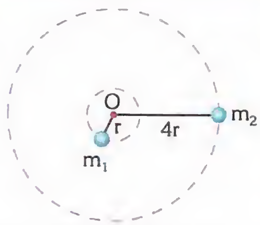



An object with a mass of 0.5 kg is at rest and 10 cm from the centre of a rotating horizontal table, as shown in the figure. When the angular speed of the table is increased, the object starts to slide at 6 rad/s.

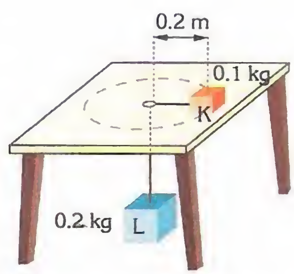
a) Calculate the centripetal force acting on the object at this speed.

b) Calculate the maximum coefficient of static friction between the object and the table.

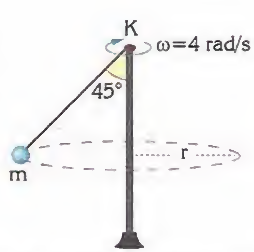
**20.** Masses $m_I$ and $m_2$ are rotating with circular trajectories of radii r and 4r on a smooth horizontal surface at the tangential speeds of $v_1 = 2$ m/s and $v_2 = 8$ m/s, as shown in the figure. What should the ratio of masses $m_1/m_2$ be for the centripetal forces $F_1$ and $F_2$ acting on the objects to be equal?



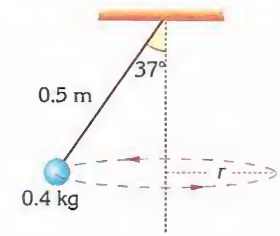

**21.** As object K with a mass of 0.1 kg performs uniform circular motion on a smooth table, as shown in the figure. Object L of mass 0.2 kg, remains stationary. If the length of the string between object K and the hole in the table is 0.2 m, what is the tangential speed of object K?




**22.** An object of mass m, attached to the end of a string, has its other end fixed to the top of a vertical rod. If this system is rotated at an angular speed of 4 rad/s, the string makes an angle of 45° with the vertical, as shown in the figure. Find the radius,r, of the circular motion. (Take g=10 N/kg)




**23.** A 0.4 kg object attached to the end of a 0.5 m long string is rotated in a horizontal plane at a constant speed as the string makes an angle of 37° with the vertical.

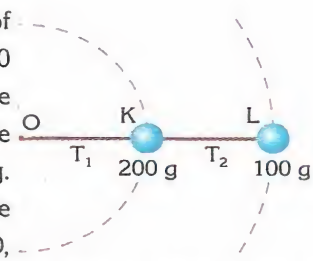



a) Calculate the angular speed of the object.

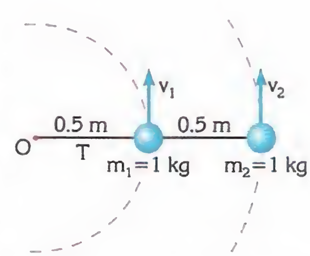
b) Calculate the centripetal acceleration of the object.

c) Find the tension in the string. (Take g=10 N/kg)

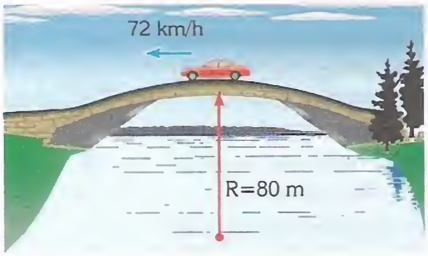
**24.** Objects K and L of masses 200 g and 100 g, respectively, are attached to the middle and the end of a string. The other end of the string is fixed at point O, and the string is rotated about point O on a horizontal frictionless surface, as shown in the figure. What is the ratio of tension $T_1$ between O-K to tension $T_2$ between K–L, $\frac{T_1}{T_2}$?




**25.** Objects of mass 1 kg each are attached to the middle and the end of a 1 m long rope. The objects then perform uniform circular motion about point O on a smooth horizontal surface. If the tension T in the figure is 54 N, what is the speed $v_1$?

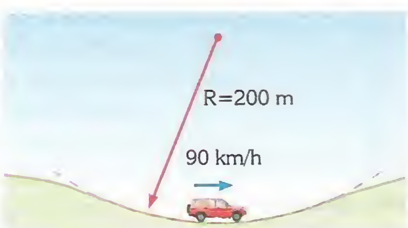


**26.**




A car having a mass of 960 kg, including the mass of the driver, is passing over a curved bridge of radius of curvature 80 m. The car has a constant speed of 72 km/h. What is the apparent weight of the car when it passes the highest point of the bridge, as shown in the figure? Does the driver at this moment feel that he is heavier or lighter?

**27.**

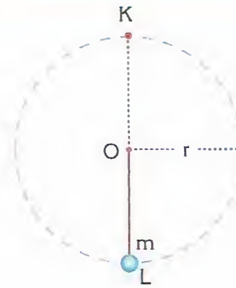



While a car of total mass 1000 kg is passing through a valley, along a road of radius of curvature 200 m, it moves at a constant speed of 90 km/h.

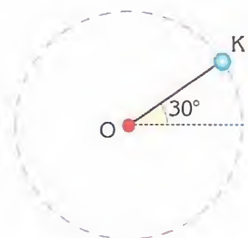
a) What is the apparent weight of the car while it passes the lowest point of the road?

b) What is the apparent weight of the driver whose mass is 80 kg?

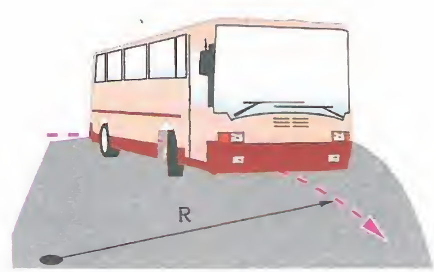
**28.** An object of mass m is attached to the end of a rope of length, r. When it is at rest, as shown in the figure, the tension in the rope is T. When the object executes uniform circular motion in the vertical plane, the tension in the rope is zero when it passes point K at the top of the circle. By how many times is the tension greater than T in the rope when the object is passing point L?




**29.** An object is exhibiting uniform circular motion in a vertical plane, as shown in the figure. When it passes point K, the tension in the string is equal to the actual weight of the object. Find the the maximum tension which will appear in the string during the motion in terms of the actual weight of the object.

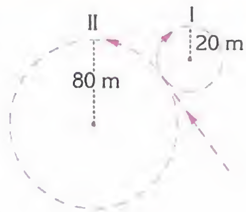



**30.**




If a bus can turn round a flat curved road, of radius 50 m, at a maximum speed of 54 km/h without skidding, what is the coefficient of static friction between the wheels and the road?

31. A car comes to a corner and can turn curve I, as shown in the figure, at a maximum speed of 10 m/s. If the curves are flat, what should its maximum speed be so that it can turn curve II safely?



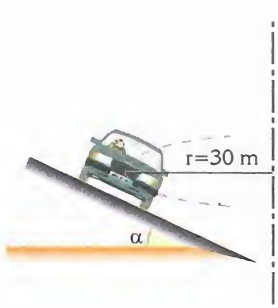

32. If a car can turn round an inclined curve of radius 30 m at a maximum speed of 54 km/h without slipping,

a) Draw a diagram to show the forces acting on the car.

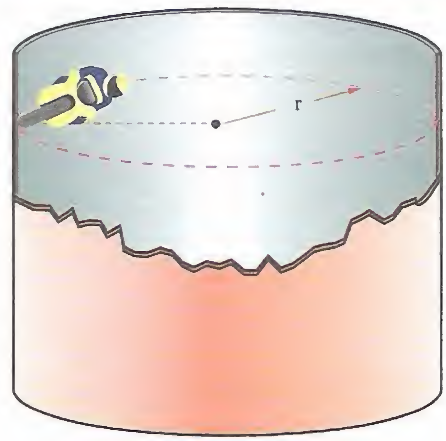
b) find the slope of the curve.

c) If the mass of the car is 1200 kg, what is the centripetal force that acts on the car?

33. A car turns round a curve of radius 30 m, on a banked road, at a speed of 20 m/s, as shown in the figure. What must the banking angle, $\alpha$ of the road be so that the car can turn round this curve without skidding?



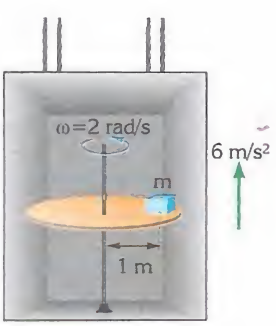

34.




A motorcyclist is performing a show on the lateral surface of a cylinder having a radius of 5 m, as shown in the figure. What must his minimum speed be so as not to fall?

(Take the coefficient of friction between the lateral surface of the cylinder and the motorbike to be 0.5.)

35. A lift is accelerated upwards at 6 $m/s^2$. An object of mass, m, inside the lift rests on a table which is rotating at an angular speed of 2 rad/s, as shown in the figure. What should the minimum value of the coefficient of static friction between the object and the table be so that the object stays on the table without sliding?




## 8.2 The Universal Law of Gravity

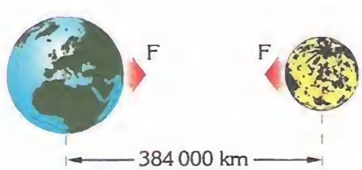
36. Why does an object thrown upwards slow down, stop and fall back downwards?

**37.** What does the attraction between masses depend upon?

**38.** Why is the weight of a body different at the poles and at the equator of the Earth?

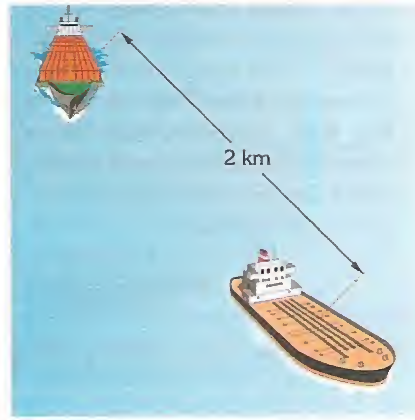
**39.** Is it true that the Earth and the Moon apply "equal and opposite" forces upon each other? Since it has a smaller mass, why does the Moon not crash into the Earth?

**40.**



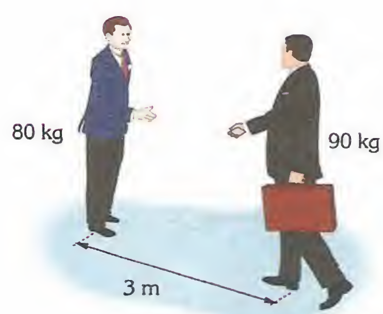

If the mass of the Earth is $6\times10^{24}$ kg, and the mass of the Moon is $7\times10^{22}$ kg and the distance between them is 384 000 km, calculate the gravitational force acting between them.
(Take $G = 6.67\times10^{-11}$ N·m²/kg²)

**41.**



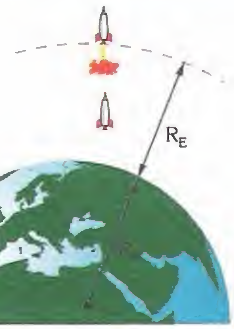

If the distance between two ships, one weighing $8\times10^5$ tons and the other $3\times10^5$ tons is 2 km, find the gravitational force between these ships.
(Take $G = 6.67\times10^{-11}$ N·m²/kg²)

**42.**




What is the gravitational force acting on two men, one weighing 80 kg and the other weighing 90 kg, who are standing 3 metres away from each other?
(Take $G = 6.67\times10^{-11}$ N·m²/kg²)

**43.**

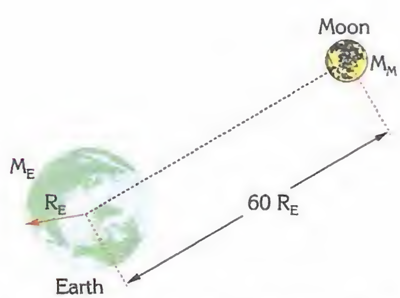



What is the ratio of the weight of a rocket of mass 1800 kg on the Earth to the gravitational force acting on it when it is at a distance equal to the Earth's radius, above the Earth?

**44.** The mass of Jupiter is $1.9\times10^{27}$ kg and its radius is $7\times10^{4}$ km, whereas, the mass of Mars is $6.4\times10^{23}$ kg and its radius is $3.4\times10^{3}$ km. Find the gravitational acceleration on the surfaces of Jupiter and Mars.

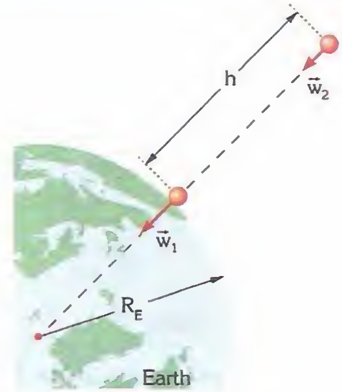
(Take $G=6.67\times10^{-11}$ N·m²/kg²)

**45.**




Where is the gravitational force acting on an object between the Earth and the Moon zero? Find the result in terms of $R_E$. (The mass of the Earth is 81 times greater than that of the Moon and the distance between their centres is approximately 60 times greater than Earth's radius.)

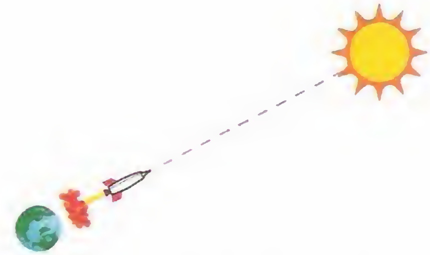
**46.**




In order to decrease the weight of an object by 10% on Earth, to what height should it be raised?

($R_E = 6.4\times10^{3}$ km)

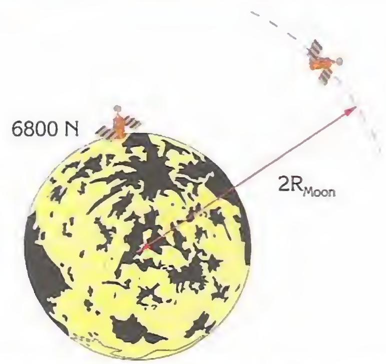
**47.**



At what distance from the Earth do the gravitational forces of the Earth and the Sun, affecting a spaceship which is travelling from Earth towards the Sun, cancel each other out? (The distance between the Earth and the Sun is $15\times10^{7}$ km;

$M_E = 6\times10^{24}$ kg; $M_S = 2\times10^{30}$ kg)

**48.**

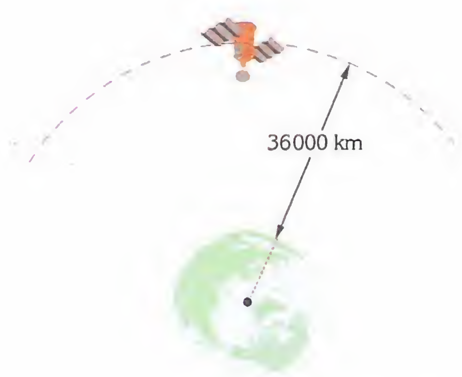



The weight of a space probe on the Moon is 6800 N.

a) If the gravitational acceleration of the Moon is approximately 1.7 N/kg, what is the mass of the probe?

b) What is the gravitational force on the probe while it is orbiting the Moon at twice the radius of the Moon from its centre?

**49.**



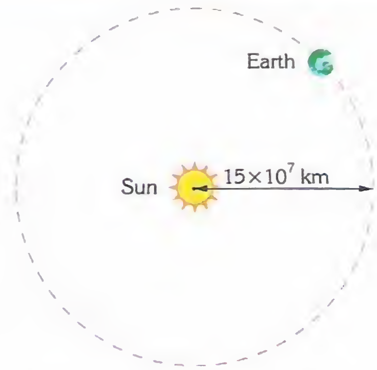

A communication satellite is orbiting the Earth 36 000 km above sea level. Knowing that the radius of the Earth is 6 400 km, its mass is $6\times10^{24}$ kg, and the universal gravitational constant is $G=6.7\times10^{-11}$ $Nm^2/kg^2$ find

a) the period of the satellite?

b) the tangential speed of the satellite?

(Take $\pi=3.14$)

**50.**



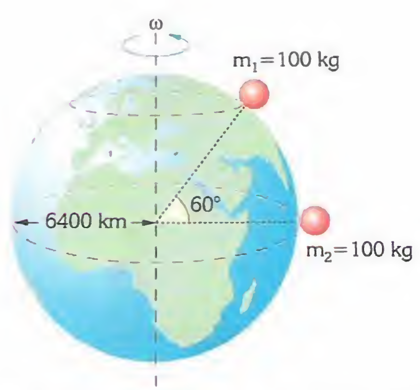

Considering the Earth is orbiting in a circular path around the Sun, as shown in the figure.

a) Calculate the speed of rotation of the Earth around the Sun.

b) Calculate the mass of the Sun.

(Take $\pi=3.14$; 1 year=365 days)

**51.**




Two objects, each of mass 100 kg, perform circular motion around the Earth. One of the objects is at the equator and the other is at a latitude of 60° (as shown in the figure). Assuming the radius of the Earth is approximately 6 400 km, find

a) their angular speeds

b) their tangential speeds

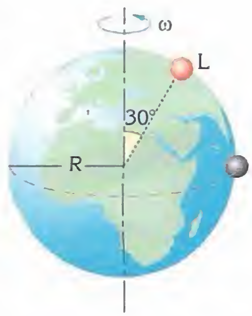
c) the difference in their apparent weights

52. a) What should the tangential speed of the Earth be for a man at the Equator to be weightless?

b) In this case, how long would a day be?

(Take the radius of the Earth to be 6 400 km; and $\pi=3.14$)

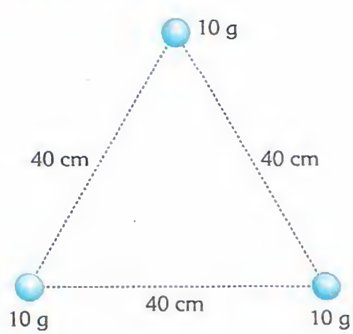
53. An object whose apparent weight is 1 N at point K on the Earth, has an apparent weight of $\frac{4}{3}$N at point L.




Assuming that the Earth is perfectly spherical and rotating about its axis at a constant angular speed ω, what is the centripetal force acting on the object at point K

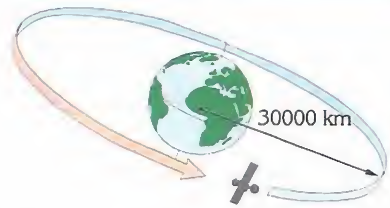
## 8.3 General Form of Gravitational Potential Energy
## 8.4 Binding Energy
## 8.5 The Escape Speed

54.




Three objects having a mass of 10 g each are placed on each corner of an equilateral triangle whose side length is 40 cm, as shown in the figure. What is the potential energy of this system?

55.




A 1000 kg satellite orbiting the Earth is 30 000 km away from the centre of the Earth. Using the data given in the next question, answer the questions below.

a) What is the potential energy of the Earth-satellite system?

b) What is the kinetic energy of the satellite?

c) What is the binding energy of the satellite?

d) What must the speed of the satellite be to escape from this orbit?

56. Calculate the binding energy and the escape speed of an object of mass $10^4$ kg which is stationary on Earth, with respect to

a) the Earth

b) Our galaxy, the Milky Way.

Take $R_E = 6.4\times10^6$ m,

$r_{E\text{-}M} = 3\times10^{20}$ m

$M_E = 6\times10^{24}$ kg,

$M_M = 8\times10^{41}$ kg,

$G = 6.67\times10^{-11}$ N·m²/kg²

CHAPTER 9

# Simple Harmonic Motion and Mechanical Waves

*Oscillation - also known as periodic motion - is an important feature of the physical world. Some of the oscillations occur in a reliable and regular way. such as the beating of the human heart and the changing of the seasons. Electrical oscillations make it possible to use television sets and radios. At the atomic level, atoms oscillate within molecules about their relatively fixed positions. The simplest form of oscillation is called simple harmonic motion. We can observe many objects that undergo simple harmonic motion in our daily life: A tuning fork, a child's swing, a guitar string, a pendulum in a clock, a bell, pistons in a car, and suspension systems.*

*In this chapter, two simple systems which undergo simple harmonic motion will be discussed: a mass-spring system and a pendulum. The basic concepts of mechanical waves will also be discussed. since oscillations and wave motion are intimately related subjects.*

## 9.1 VIBRATIONS AND SPRINGS

Figure 9.1 shows the stable equilibrium of a ball which remains at the bottom of a bowl. A slight disturbance will cause its equilibrium position to be broken and to start it swinging. The swinging motion will stop after a certain time and the previous equilibrium position will be regained. Thus, if a body returns to its original position after being subjected to a force, it is said to be in **stable equilibrium.** It is the component of the ball's weight, $\vec{w}_{//}$, parallel to the surface, that brings the ball back to the equilibrium position. This force is called the **restoring force**. In the absence of opposing, or frictional forces, the motion about the equilibrium position continues. This motion is said to be a **mechanical vibration or oscillation.**

The most suitable physical system for analysing oscillations is a mass-spring system, as shown in Figure 9.2.a

When the mass is displaced by an external force from its equilibrium position, as shown in Figure 9.2.b, the spring exerts a force $\vec{F}_s$ which returns it to its equilibrium position. Here, $\vec{F}_s$ is the restoring force and it obeys the equation $\vec{F}=-k\vec{x}$, which was explained in Section 6.6.

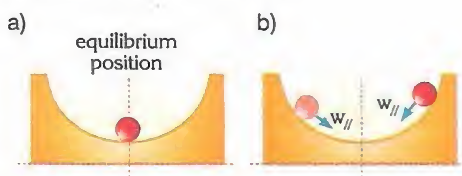



**Figure 9.1** a) A ball in stable equilibrium b) when it is displaced from its equilibrium position, the component of its weight parallel to the surface brings it back to the equilibrium position.

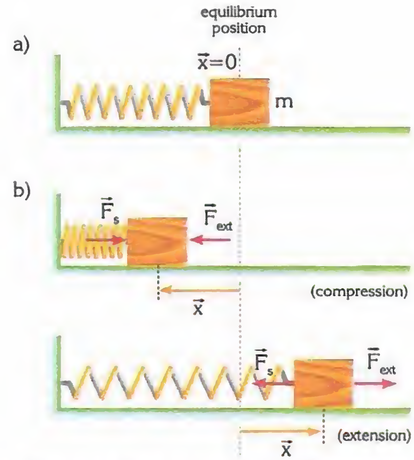



**Figure 9.2** a) A mass, m, attached to a spring rests on a frictionless surface.

b) When mass m is displaced by an amount $\vec{x}$, (compression or extension) the spring exerts a restoring force $\vec{F}_S$ in a direction opposite to $\vec{x}$ which brings it back to the equilibrium position.

### Example 9.1 — Restoring force of a spring

A dynamometer has a spring constant of 100 N/m. Calculate the displacement (amount of stretching) of the spring if it is used to measure the weight of

a) a mass of 400 g,

b) a mass of 10 kg. (Take g = 10 N/kg)

**Solution**

a) When a load of 400 g is hung from the dynamometer, its weight is balanced by the spring force.

$$w = F_s$$
$$mg = kx$$
$$(0{,}4\ \text{kg})(10\ \text{N/kg}) = (100\ \text{N/m})x$$
$$x = 0.04\ \text{m} = 4\ \text{cm}$$

b) As in part a), the 10 kg load can be balanced by the spring force.

$$w = F_s$$
$$mg = kx$$
$$(10\ \text{kg})(10\ \text{N/kg}) = (100\ \text{n/m})x$$
$$x = 1\ \text{m}$$

We should expect that this much displacement is above the elastic limit and such a load would break the dynamometer.

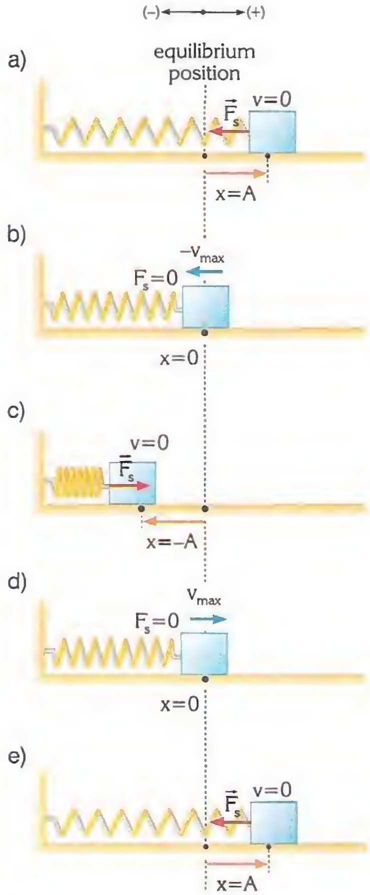



**Figure 9.3** Spring force on, and velocity of, a mass at different positions of its oscillation.

# 9.2 SIMPLE HARMONIC MOTION

## a. Basic Definitions of Simple Harmonic Motion

Any oscillating system for which the restoring force is directly proportional to the negative of the displacement ($\vec{F}= -k\vec{x}$ as explained in Section 6.6) is said to undergo **simple harmonic motion**. In this section the simple harmonic motion of a mass-spring system placed on a horizontal frictionless surface will be analysed. The mass-spring system is the simplest system which can perform simple harmonic motion obeying the equation $\vec{F}=-k\vec{x}$.

The spring is initially stretched a distance of x=A, as shown in Figure 9.3.a, and then released. The restoring force $\vec{F}_s$ pulls it towards the equilibrium position. Therefore, the mass accelerates and passes the equilibrium position with maximum speed. Indeed, as the mass moves towards the equilibrium position, the force acting on it, and thus its acceleration, decreases to zero, as shown in Figure 9.3.b. After it moves to the left of the equilibrium position, the restoring force slows the mass down. The mass stops momentarily at the maximum compression where x =−A, as shown in Figure 9.3.c. It then begins moving back in the opposite direction, as shown in Figure 9.3.d, until it reaches the starting point, x=A, as shown in Figure 9.3.e. It then repeats the motion, moving back and forth symmetrically between x=A and x=−A.

To analyse simple harmonic motion, the following terms will be defined:

**Displacement (x):** The position of the mass from the equilibrium position at a given moment. Its unit is the metre.

**Amplitude (A):** The maximum displacement from the equilibrium position.

A **cycle** is a complete oscillation. For example, when the mass starts at x=A, as in Figure 9.3.a, and returns to the the same position, as in Figure 9.3.e, it will have completed a cycle (one complete round trip).

**Period (T):** The time required for one complete cycle. Its SI unit is the second.

**Frequency (f):** The number of complete cycles per second. Frequency is generally specified in hertz (Hz), where 1 Hz = 1 cycle per second. The relationship between period and frequency can be given by the equations

$$f = \frac{1}{T} \quad \text{or} \quad T = \frac{1}{f}$$

for example, if the frequency is 5 cycles per second, then each cycle takes 0.2 s.

## b. Deriving the Equations of Simple Harmonic Motion

The relationship between uniform circular motion and simple harmonic motion can be used to derive the equations of simple harmonic motion. As an object performs uniform circular motion, its projection (or shadow) on a screen produces simple harmonic motion. In Figure 9.4, as a toy car goes around a circular path, its shadow on the screen appears to oscillate back and forth like the simple harmonic motion of a mass on a spring. Here, if the car starts its motion at $x=+A$, its simple harmonic motion projection onto the screen starts from the maximum displacement. In Figure 9.5, it is assumed that P is a point on the car moving at a constant speed v in a circle of radius A. The projection (or shadow) of P moves back and forth between $x=+A$ and $x=-A$.

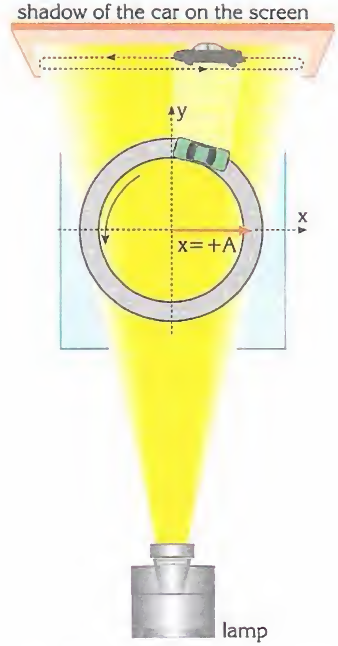



Figure 9.4 A possible experimental set-up illustrating the connection between uniform circular motion and simple harmonic motion. As the car goes around a circular path, it casts a shadow on the screen that moves with simple harmonic motion.

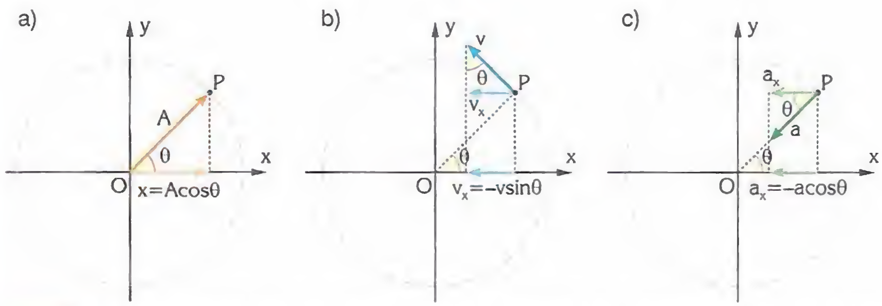



Figure 9.5 Analysis of a) the position b) the velocity and c) the acceleration of the projection of the toy car on the x axis.

From Figure 9.5.a, the position of point P along the x-axis is

$$x = A\cos\theta$$

since $\cos\theta$ oscillates between $+1$ and $-1$, A is the maximum displacement (amplitude) of the projection of point P along the x-axis.

Since $\theta = \omega t = \frac{2\pi t}{T}$, the equation above can be expressed as

$$x = A\cos\omega t$$

where $\omega$ is the angular speed of circular motion. It also represents the **angular frequency** of simple harmonic motion because the period, T, is the same for both circular motion and simple harmonic motion.

From Figure 9.5.b, the velocity of the projection onto the x axis, $v_x$, can also be expressed as a function of time, we find that it is proportional to $-\sin\theta$

$$v_x = -v\sin\theta \quad \text{where} \quad \theta = \omega t \quad \text{so} \quad v_x = -v\sin\omega t$$

where v is the tangential speed of circular motion and the maximum velocity of the projection on the x-axis. The negative sign occurs because $v_x$ is directed towards the negative x-axis.

From Section 8.1.a, the tangential speed of uniform circular motion is

$v = \omega r$ here $r = A$ then we can write

$v = \omega A$

This equation also gives the maximum velocity of the projection performing simple harmonic motion. This is because in the equation $v_x = -v\sin\omega t$, thus, $v_x$ reaches its maximum value when $\sin\omega t = 1$,

$v_x = v_{max}$ $v_{max} = -A\omega$ (maximum velocity in simple harmonic motion)

So the velocity equation can be re-written as

$$v_x = -A\omega \sin \omega t$$

In Chapter 8, it was shown that any object moving in a circle with a constant speed accelerates toward the centre. As shown in Figure 9.5.c, the acceleration of the projection onto the x axis, $a_x$, is proportional to $-\cos\theta$ and can be expressed as a function of time as follows;

$$a_x = -a\cos \omega t$$

where a is the centripetal acceleration of the circular motion and the maximum acceleration of the projection onto the x-axis. As in the velocity equation, the negative sign occurs because $a_x$ is directed towards the negative x-axis.

In the equation $a_x = -a\cos\omega t$, a is the centripetal acceleration of the uniform circular motion. From Section 8.1.b

$a = A\omega^2$

This acceleration represents the maximum acceleration of an object performing simple harmonic motion, since in the equation $a_x = -a\cos\omega t$, $a_x$ reaches its maximum value when $\cos\omega t = 1$. That is,

$a_x = a_{max}$ $a_{max} = A\omega^2$ (maximum acceleration of simple harmonic motion)

The equation for acceleration can be re-written as

$$a_x = -A\omega^2 \cos \omega t$$

In Figure 9.6.a comparison between the simple harmonic motion of a mass-spring system and the projection of an object's uniform circular motion onto the x axis is shown.

**Important note:** The equations above (for x, v, and a) are valid for simple harmonic motion of a system in any plane, provided that it starts from maximum displacement.

If a system starts its simple harmonic motion from its equilibrium position towards the positive x direction, the equations below are applied, because when t=0, these equations must give the expected maximum or minimum values

$x = A\sin\omega t$

$v = A\omega\cos\omega t$

$a = -A\omega^2\sin\omega t$

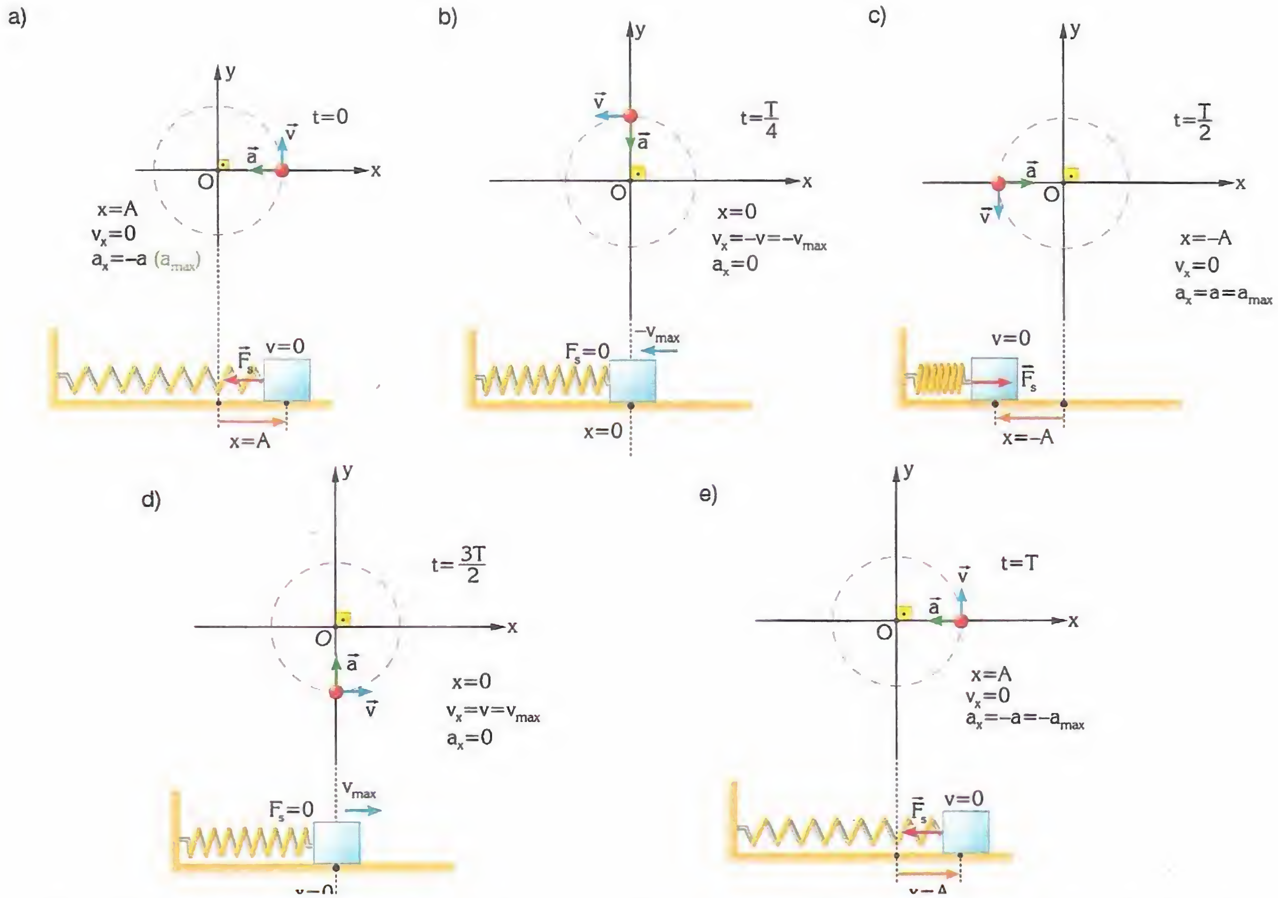



Figure 9.6 A comparison between the simple harmonic motion of a mass-spring system and the projection of an object's uniform circular motion onto the x axis.

## Simple Harmonic Motion Graphs

An object attached to a spring oscillates. How do the displacement, velocity and acceleration of the object change during the oscillation?

In Figure 9.7, an object is oscillating vertically at the end of a spring. A pencil attached to the object, and moving along the y-axis, marks the graph paper, which is moving at a constant speed. Thus, a displacement-time graph of the oscillating object is obtained.

The horizontal axis on the graph paper represents the time axis and the vertical axis represents the displacement. If the motion starts from the equilibrium position and moves towards the +y axis (ignoring any external effects and friction), a y-t graph will be obtained, as shown in Figure 9.8.a. The equations of motion for the simple harmonic motion of the spring are

$$y = A\sin(\omega t)$$

$$v = \omega A\cos(\omega t)$$

$$a = -\omega^2 A\sin(\omega t)$$

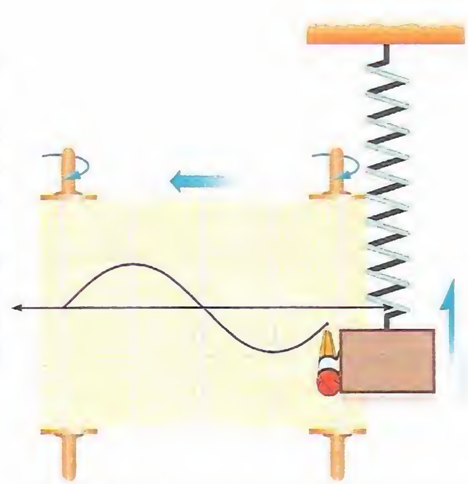


Figure 9.7 A mass-spring system set-up to examine simple harmonic motion.

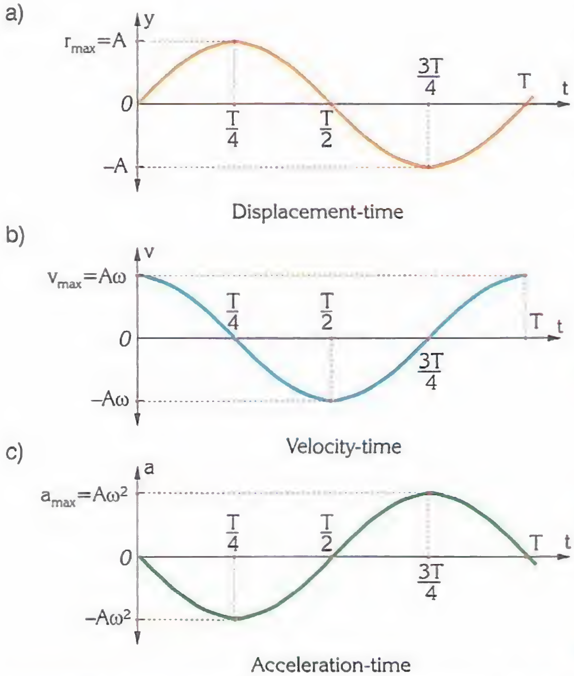



**Figure 9.8** a) Displacement-time, b) velocity-time, and c) acceleration-time graphs of simple harmonic motion, which starts from the equilibrium position and commences towards the positive y axis.

From the equations of the velocity and the acceleration, the velocity-time and acceleration-time graphs can be plotted, as shown in Figure 9.8.b and c

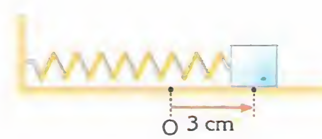
When the mass, m of the attached load or the spring constant, k in the system shown in Figure 9.7 is changed, the y-t, v-t, and a-t graph characteristics of each oscillation will also change. However, all of the graphs will commonly exhibit a repetitive oscillation over a fixed time (between two defined points).

From the equations and graphs of simple harmonic motion in Figure 9.8, the following can be stated:

1. The velocity of simple harmonic motion decreases as the object moves away from equilibrium and is zero at the maximum displacement. While the object is passing through the equilibrium position the velocity is at its maximum rate.
2. The acceleration of simple harmonic motion increases as the object moves away from equilibrium and is maximum at the maximum displacement. While the object is passing through the equilibrium position, the acceleration is zero.
3. The acceleration is proportional to the displacement but has the opposite direction, as the displacement increases the acceleration increases and as the displacement decreases the acceleration decreases.

## Example 9.2 Equations of simple harmonic motion

A mass-spring system which is stretched 3 cm and then released performs simple harmonic motion with an angular speed of $2\pi$ rad/s. State the




a) amplitude,
b) displacement equation,
c) velocity equation,
d) acceleration equation.
e) What is the displacement, velocity and acceleration when $t = \frac{2}{3}$ s ? (Take $\pi = 3$)

**Solution**

a) Since the system is stretched 3 cm, the amplitude (maximum displacement) of its simple harmonic motion will be 3 cm.

b) Since the system starts its motion from maximum displacement in the positive x direction, the equation for the displacement is expressed as

$$x = A\cos\omega t$$

inserting the values $A = 3$ cm and $\omega = 2\pi$ rad/s into the equation,

$$x = 3\cos(2\pi t) \text{ cm}$$

c) The equation for the velocity is

$$v = -\omega A\sin\omega t$$

inserting the values $A = 3$ cm and $\omega = 2\pi$ rad/s into the equation,

$$v = -6\pi\sin(2\pi t) \text{ cm/s}$$

d) The equation for the acceleration is

$a = -\omega^2 A \cos \omega t$

inserting the values A=3 cm and $\omega=2\pi$ rad/s into the equation,

$a = -12\pi^2 \cos(2\pi t)$ cm/s$^2$

e) $x = A \cdot \cos \omega t = (3\text{ cm}) \cos 2\pi \cdot \frac{2}{3} = 3 \cdot \cos \frac{4}{3}\pi$

$x = -3 \cdot \cos 60° = -\frac{3}{2}$ cm

$v = -A\omega \sin \omega t = -(3\text{ cm})2(3\text{ rad/s}) \sin \frac{4}{3}\pi$

$v = (18\text{ cm/s}) \sin 60 = (18\text{ cm/s}) \frac{\sqrt{3}}{2} = 9\sqrt{3}$ cm/s

$a = -A\omega^2 \cos \omega t = -(3\text{ cm})(36\text{ rad}^2/\text{s}^2) \cos \frac{4}{3}\pi$

$a = (108\text{ cm/s}^2) \cos 60 = 54\text{ cm/s}^2$

## Example 9.3

Equations in simple harmonic motion

If a mass-spring system starts its simple harmonic motion from the equilibrium position towards the +x axis and has an angular speed, $2\pi$ rad/s and an amplitude 3 cm, write down the equations for the displacement, velocity and acceleration of the system.

**Solution**

Since it starts from its equilibrium position, the equations are expressed as follows

$x = A \cdot \sin \omega t = 3(\sin 2\pi t)$ cm

$v = A\omega \cos \omega t = 6\pi(\cos 2\pi t)$ cm/s

$a = -A\omega^2 \sin \omega t = -12\pi^2(\sin 2\pi t)$ cm/s$^2$

## Example 9.4

Equations of simple harmonic motion

An object is exhibiting simple harmonic motion obeying the equation $x=2 \sin(100\pi t)$ m

a) Where does this system begin its motion?

b) Calculate the amplitude, angular speed, period and frequency of the simple harmonic motion.

c) Find the maximum speed and the maximum acceleration of the object.

**Solution**

a) When t=0, sin 0=0 so x=0. That is, it starts its motion from the equilibrium position towards the +x axis.

b) The displacement equation for simple harmonic motion is $x = A \sin(\omega t)$

Since it is given as $x = 2 \sin(100\pi t)$

Thus the amplitude is $A = 2$ m

and the angular speed is $\omega = 100\pi$ rad/s

From $\omega = \frac{2\pi}{T}$, the period is $T = \frac{1}{50}$ s

From $T \cdot f = 1$, the frequency is $\left(\frac{1}{50}\text{ s}\right) f = 1$

so $f = 50\text{ s}^{-1}$

c) The maximum speed is given by

$v = \omega A = (100\pi\text{ rad/s})(2\text{ m}) = 628$ m/s

The equation for maximum acceleration is

$a = \omega^2 A = (100\pi\text{ rad/s})^2(2\text{ m}) = 19.8\times10^4$ m/s$^2$

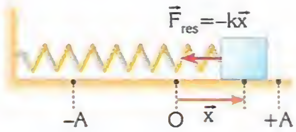



**Figure 9.9** A mass-spring system. Equilibrium position is at O and the amplitude of simple harmonic motion is A.

## 9.3 SIMPLE HARMONIC MOTION OF A MASS-SPRING SYSTEM

### a. Period of a Mass-Spring System

We know that when the mass of a mass-spring system is stretched a distance A and then released, the mass exhibits simple harmonic motion, accelerating between the positions +A and –A (here, A also refers to the amplitude) in a frictionless environment, as shown in Figure 9.9. In Section 9.2 the relationship between uniform circular motion and simple harmonic motion was analysed and the following equations derived for simple harmonic motion

$$x = A\cos\omega t \qquad v = -\omega^2 A \sin\omega t \qquad a = -\omega^2 A \cos\omega t$$

From these equations, a relationship between the equations of displacement x and acceleration, a, can be derived as

$$a = -\omega^2 x$$

This acceleration is provided only by the restoring force of the spring, $\vec{F}_{res} = -k\vec{x}$, which was explained in Section 9.1. From the second law of motion

$$\vec{F}_{net} = m\vec{a}$$

$$-kx = -m\omega^2 x$$

$$\omega = \sqrt{\frac{k}{m}}$$

Equating the last equation to $\omega = \frac{2\pi}{T}$, an expression for the period, T, of the system is obtained as

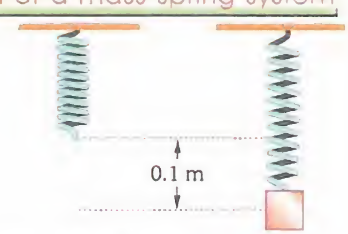
$$T = 2\pi\sqrt{\frac{m}{k}}$$

Note that, the period of a mass-spring system is independent of the amplitude of motion and the gravitational acceleration.

## Example 9.5

Period of a mass-spring system

When an object of mass 2 kg is attached to the end of a spring, the spring stretches by a distance of 0.1 m and reaches equilibrium. Calculate the period of the oscillation that will occur if it is stretched more and then released.
(Take $g = 10\ m/s^2$ and $\pi = 3$)




### Solution

In equilibrium, the weight of the object is balanced by the restoring force (kx) that the spring exerts on the object. This is

$$kx = mg$$

When the values are placed in the equation

$$k\,(0.1\ m) = (2\ kg)(10\ N/kg) \quad \text{then} \quad k = 200\ N/m$$

Then the period will be

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\cdot 3\sqrt{\frac{2\ kg}{200\ N/m}} \quad \text{thus} \quad T = 0.6\ s$$

## b. Springs Connected in Series and Parallel

A mass-spring system can consist of more than one spring. In this case, the effective value of the spring constant of the system, called the **equivalent spring constant**, can be found according to the configuration in which the springs are attached. If springs are attached end-to-end, as shown in Figure 9.10.a, it is said that these springs are connected in series. If springs are attached side by side, as shown in Figure 9.11.a, they are said to be in parallel. The equivalent spring constant of springs attached in series can be found as follows.

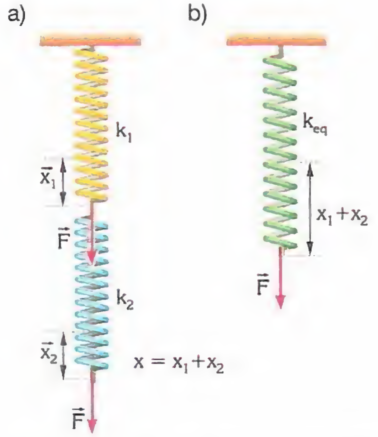



**Figure 9.10** a) Two springs connected in series, b) A single spring undergoing the same amount of extension as the spring-system in (a) under the effect of the same force.

Assume that two springs with spring constants $k_1$ and $k_2$ are attached in series, as shown in Figure 9.10.a. When the lower spring is stretched down with a force $\vec{F}$, this force will act equally on both of the springs, but since the spring constants are different, the displacements of the springs will be different. This is expressed as follows:

$$\vec{F} = k_1\vec{x}_1 \quad \text{and} \quad \vec{F} = k_2\vec{x}_2$$

The two springs can be replaced with a single spring, which undergoes the same amount of extension, $\vec{x}_1 + \vec{x}_2$, as the two springs in series, when subjected to the same external force $\vec{F}$, as shown in Figure 9.10.b.

The spring constant of the single spring has an equivalent spring constant, $k_{eq}$, to the two-spring system in series.

Thus, Hooke's law for the single spring can be expressed as

$$\vec{F} = k_{eq}(\vec{x}_1 + \vec{x}_2)$$

If we substitute $F/k_1$ for $x_1$ and $F/k_2$ for $x_2$ into the equation above, and perform some manipulation, the following expression for the equivalent spring constant, $k_{eq}$ is obtained

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

In the same way the equivalent spring constant for two springs connected in parallel, as shown in Figure 9.11.a can be calculated.

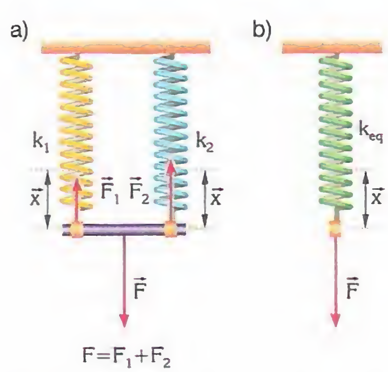



**Figure 9.11** a) Two springs connected in parallel, b) A single spring undergoing the same amount of extension as the spring-system in (a) under the effect of the same force.

When the two springs are extended by a force $\vec{F}$, both of the springs are stretched downwards a distance x. Since the spring constants of springs $k_1$ and $k_2$ are different, the force acting on each spring will be different.

$$\vec{F}_1 = k_1\vec{x} \quad \text{and} \quad \vec{F}_2 = k_2\vec{x}$$

The two springs can be replaced with a single spring, which undergoes the same amount of extension, $\vec{x}$, as the system of two parallel springs, under the same external force F, as shown in Figure 9.11.b. The spring constant of this single spring has an equivalent spring constant, $k_{eq}$ to the two-spring system in parallel.

The sum of the spring forces $F_1$ and $F_2$ is equal to the external force F. This can be expressed as

$F = F_1 + F_2$

If these forces are written separately,

$F = k_{eq}\,x$ , $F_1 = k_1\,x$ and $F_2 = k_2\,x$

From these equations, if $k_{eq}x$ for F, $k_1x$ for $F_1$ , and $k_2x$ for $F_2$ is substituted into the force summation equation above,

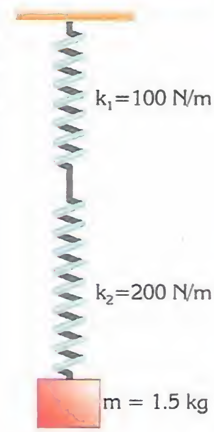
$k_{eq}\,x = k_1\,x + k_2\,x$

Thus, the equivalent spring constant, $k_{eq}$, of a system with two springs connected in parallel is

$$k_{eq} = k_1 + k_2$$

## Example 9.6

Period of springs in series




When an object of mass 1.5 kg is attached to the end of two springs in series, with spring constants $k_1$=100 N/m and $k_2$=200 N/m, the system comes to an equilibrium, as shown in the figure. Calculate the period of the simple harmonic motion that will occur when the object is released after extending it. (Take $\pi$=3)

### Solution

We can apply the equation that gives the equivalent spring constant of springs in series.

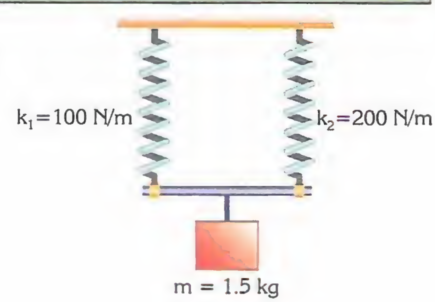
$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} += \frac{1}{100\ \text{N/m}} + \frac{1}{200\ \text{N/m}} \quad \text{thus} \quad k_{eq} = \frac{200}{3}\ \text{N/m}$$

When we substitute this value into the equation for the period of the springs

$$T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\cdot 3\sqrt{\frac{1.5\ \text{kg}}{\frac{200\ \text{N/m}}{3}}} = 0.9\ \text{s}$$

## Example 9.7

Period of springs in parallel




An object of mass 1.5 kg is attached to two springs in parallel, with spring constants $k_1$=100 N/m and $k_2$=200 N/m, as shown in the figure. Calculate the period of the simple harmonic motion that will occur after extending and releasing the object. (Take $\pi$=3)

### Solution

Applying the equation for the equivalent spring constant of springs in parallel

$$k_{eq} = k_1 + k_2 = 100\ \text{N/m} + 200\ \text{N/m} = 300\ \text{N/m}$$

When we substitute this value in the equation for the period of springs in parallel

$$T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\cdot 3\sqrt{\frac{1.5\ \text{kg}}{300\ \text{N/m}}} \quad \text{thus} \quad t \cong 0.42\ \text{s}$$

## 9.4 SIMPLE PENDULUM

A simple pendulum consists of a small mass, called a bob, suspended by a light and non-elastic string of length L, attached to a rigid support. When the mass is pulled to one side of its equilibrium position through a small angle $\theta$ and released, as shown in Figure 9.12, the simple pendulum performs simple harmonic motion. A child on a swing and a grandfather clock are some examples of simple pendulums. The simple harmonic motion of a simple pendulum will be analysed and an expression found for its period.

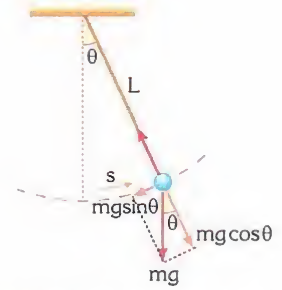



**Figure 9.12** The restoring force acting on the simple pendulum is the component of the weight, which is a tangent to the orbit. In a simple pendulum, when the angle $\theta$ is small, the magnitude of the displacement, s, is approximately equal to $L \sin\theta$.

As the simple pendulum performs simple harmonic motion, the path followed by the bob is not a straight line, but the arc of a circle whose radius equals the length, L, of the pendulum. Also, during the motion of the bob, the gravitational force acting on its mass has a component along the string and another component perpendicular to the string. The perpendicular component provides the restoring force.

$$F_{res} = -mg\sin\theta$$

The negative sign indicates that the restoring force, $\vec{F}_{res}$, is in the opposite direction to the displacement, $\vec{s}$.

The displacement, $\vec{s}$, from the equilibrium position is the length along the arc through which the mass swings. It is given by

$$s = L\theta$$

When the angle $\theta$ is sufficiently small, $\sin\theta$ may be approximated by the angle $\theta$ in radians.

$$\sin\theta \approx \theta$$

The restoring force is proportional to displacement through the equation

$$F = -mg\theta = -\left(\frac{mg}{L}\right)s$$

This equation has the same form as the mass-spring equation, $\vec{F} = -k\vec{s}$, except that here, spring constant k is replaced by mg/L. Thus, if mg/L is substituted for k in the equation for the period of the mass-spring system

$$T = 2\pi\sqrt{\frac{m}{k}}$$

The following expression is obtained for the period of the simple pendulum

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Notice that the period of the simple pendulum does not depend on the mass.

**Figure 9.13** A grandfather clock's pendulum is an example of simple harmonic motion.

## Example 9.8

The period of a simple pendulum

As illustrated in the figure, a simple pendulum of length 40 cm is oscillating over a small displacement about its equilibrium position.

a) What is the period of the pendulum?

b) If the pendulum starts oscillating from point A, at which point will it be 6 s later?

(Take $g=10\ m/s^2$, $\pi=3$)


### Solution

a) If we substitute the values $L = 40\ cm = 0.4\ m$ and $g=10\ m/s^2$ in the equation for the period,

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\cdot 3\sqrt{\frac{0.4\ m}{10\ m/s^2}}$$

$$T = 1.2\ s$$

b) If the pendulum starts oscillating from point A, since its period is 1.2 s, every 1.2 s later it will return to point A. Since 6 is an exact multiple of 1.2, in the 6$^{th}$ second, the object will again be at point A.

## Example 9.9

The period of a simple pendulum

A simple pendulum has a string of length 0.75 m and a period of 1 s. How long should the string of the pendulum be, for its period to be 2 s, at the same location?

### Solution

For the first case the period of the pendulum is;

$$T_1 = 2\pi\sqrt{\frac{L_1}{g}}$$

and for the second case $T_2 = 2\pi\sqrt{\dfrac{L_2}{g}}$

Dividing the first equation by the second one,

$$\frac{T_1}{T_2} = \sqrt{\frac{L_1}{L_2}}$$

Substituting the known values into the equation

$$\frac{1}{2} = \sqrt{\frac{0.75\ m}{L_2}}$$

Thus the string length required is found

$$L_2 = 3\ m$$

## 9.5 MECHANICAL WAVES

### a. What is a Wave?

When a pebble is thrown in a pond, some waves are produced at the point where the pebble strikes the water, as shown in Figure 9.14. These waves are circular and expand until they reach the shore.

The phenomenon which causes the surface to appear as several expanding circles will now be examined.

Figure 9.15 shows a ball on a motionless surface of a pond and a pebble being thrown onto this surface. At the instant the pebble strikes the water surface, the formation and propagation of several circles on the surface is observed. This is because some part of the kinetic energy of the falling pebble is transferred to the water. When the expanding circles reach the ball on the surface, the ball moves up and down without any net change in its position.The motion of the ball shows that it does not undergo any net displacement, since water does not flow in the direction of the disturbance. The vertical motion of the ball indicates that a temporary energy exchange occurs between the ball and the water.

This observation in fact involves two crucial phenomena. The first is that, water does not flow during the motion of the expanding circles.The second is that, the energy taken from the pebble is carried away by the expanding circles on the surface of the water. This means that water itself is not transferred from one place to another, only the energy taken from the pebble is transferred.

In general a wave is a disturbance which transmits energy from one point to another via a vibrational mechanism.

This phenomenon also occurs in a spring. The disturbance produced in a spring propagates as a wave, as shown in Figure 9.16. Therefore, water and springs are appropriate mediums in which to observe wave motion.

**Figure 9.14** A pebble thrown in a pond, produces waves at the point where it strikes the water.

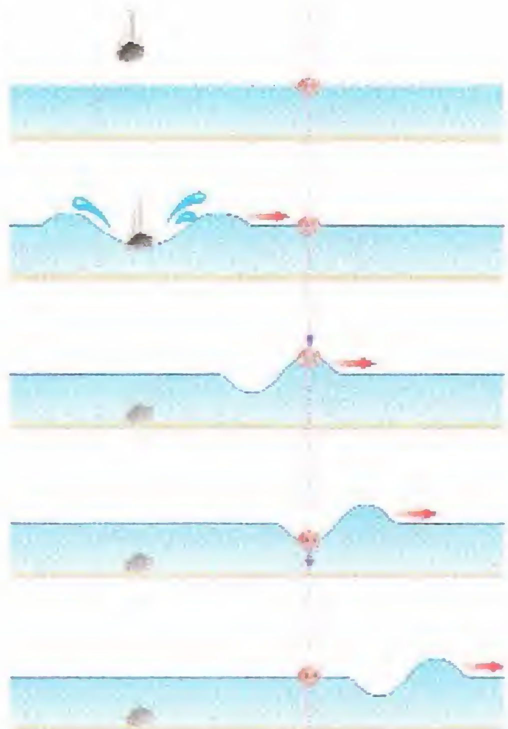


**Figure 9.15** When a pebble strikes a water surface it transfers some part of its energy to the water, causing a change in the shape of the surface. This disturbance carries the energy of the pebble from one point to another.

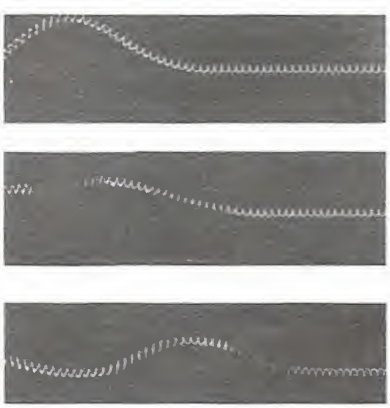


**Figure 9.16** When a spring on a smooth surface is pulled at one end and then released, the disturbance propagates as a wave. In other words, the disturbance carries the energy given to the spring along the spring in the form of a wave.

## b.Types of Waves

Waves permeate the Earth. There are mainly two types of waves: Mechanical waves and electromagnetic waves. Mechanical waves (such as water waves), waves on a spring and sound waves require a medium. The molecules of the media, in which these mechanical waves propagate, oscillate in order to transmit energy from one location to another. Electromagnetic waves are a special class of waves which do not need a medium to propagate. Visible light, radio waves, television signals, and x-rays are some examples of electromagnetic waves.

In general there are two types of mechanical waves:

**A transverse wave** is one in which the particles of a medium oscillate perpendicular to the direction of propagation. For example, when a spring is disturbed perpendicular to its length, as shown in Figure 9.17, the oscillation of an arbitrary point on the spring is perpendicular to the direction of propagation. So a wave whose direction is perpendicular to the direction of oscillation of particles is produced.

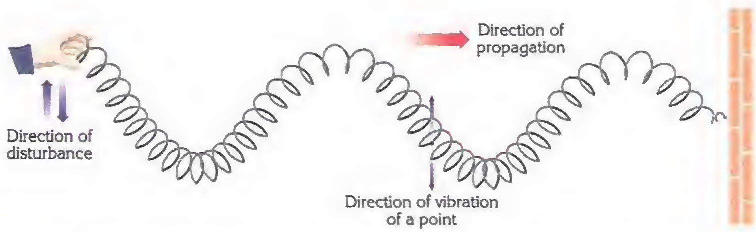



Figure 9.17 The formation of a transverse wave and the direction of oscillation of an arbitrary point on the spring

**A longitudinal wave** is one in which the particles of a medium oscillate in a direction parallel to the direction of propagation of the wave. For example, when a spring is disturbed in a direction parallel to its length, as shown in Figure 9.18, the oscillation direction of an arbitrary point on the spring is parallel to the direction of propagation of the wave. Sound waves are longitudinal waves.

Some waves have both transverse and longitudinal components such as water waves.

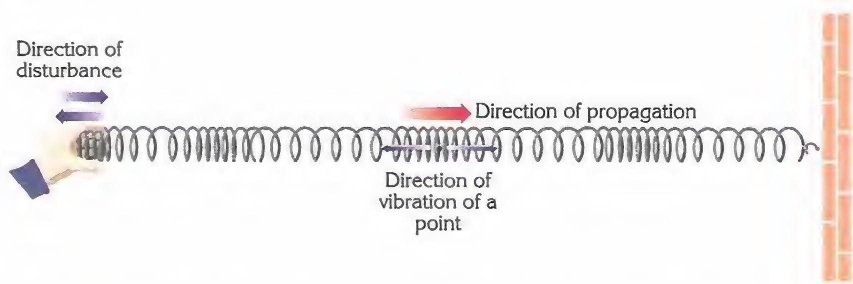



Figure 9.18 The formation of a longitudinal wave and the direction of oscillation of an arbitrary point on the spring

## c. Periodic Waves

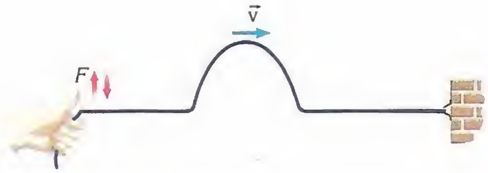



Figure 9.19 A hand flicks the string up and down once and produces a transverse pulse

A wave pulse can be produced if one end of a stretched string is flicked up and down just once, as shown in Figure 9.19. The transverse force acting on the string produces a single pulse that travels along the length of the string with a velocity $\vec{v}$. The tension in the string restores its straight line shape after the pulse has passed.

When a repetitive (periodic) motion is transmitted to the free end of the string, each particle in the string also experiences periodic motion as the wave propagates and a periodic wave is obtained. Since it is particularly easy to examine periodic waves produced by a harmonically oscillating source, an experimental arrangement can be set up to observe such waves, as shown in Figure 9.20. As the mass attached to the spring performs a simple harmonic motion, it produces a periodic wave, also called a sinusoidal wave, on the string.

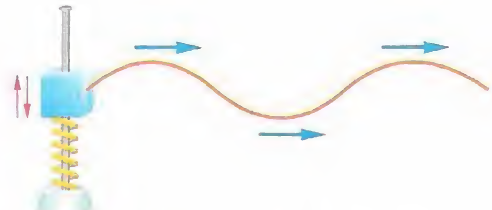


Figure 9.20 A simple arrangement illustrating the production of periodic waves by means of the simple harmonic motion of a mass-spring system.

In order to characterise a wave, some of its physical characteristics must be known: period, frequency, amplitude, wavelength, and wave speed.

During one full cycle of simple harmonic motion in the mass-spring system of Figure 9.21, one wave pattern is produced.

**Period** (T) is the time required to produce one cycle. That is, one wave pattern is produced in one period. Its SI unit is the second (s).

**Frequency** (f) is the number of cycles and therefore wave patterns produced per unit time. Its unit is the Hertz (Hz).

**Amplitude** (A) is the maximum displacement of a point on the spring from its equilibrium position. Its unit is the metre (m).

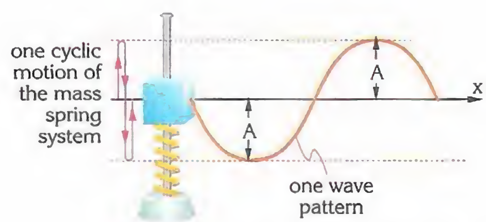



Figure 9.21 One cyclic motion of a mass-spring system in the vertical direction produces a wave pattern in the spring along the horizontal axis

**One wavelength** ($\lambda$; Greek lower case letter lambda) is the minimum distance between any two points on a wave that behave identically. For example the distance between two adjacent crests or two adjacent troughs, as shown in Figure 9.22, is one wavelength $\lambda$. Its unit is the metre (m).

**Wave speed** (v) is the distance travelled by a the wave pattern in one period T. This distance must be equal to the wavelength of the wave, $\lambda$. Its unit is metre/second (m/s). The wave speed is given by

$$v = \frac{x}{t} \qquad v = \frac{\lambda}{T} \quad \text{or} \quad v = \lambda f$$

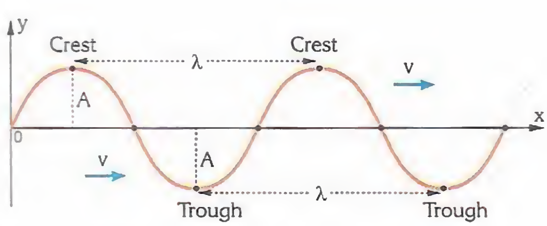



Figure 9.22 The distance between adjacent crests or adjacent troughs is one wavelength $\lambda$.

## Example 9.10 Echolocation ability of bats

A sophisticated echo location ability given to bats permits them to navigate and find the location of their prey, even in pitch darkness, by the use of sound waves. When a bat emits sound waves, they strike the prey and are reflected back to the bat. Using the time interval, $\Delta t$, between sending and receiving the sound wave, the bat can locate its prey due to the equation $x=v\Delta t$.

If the frequency of the sound waves emitted by a bat is 85 kHz and the speed of sound is 340 m/s in air, for the time interval $\Delta t=0{,}01$ s, find

a) the wavelength of sound waves emitted by the bat

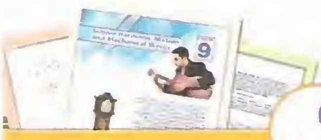
b) the distance of the prey from the bat

**Solution**

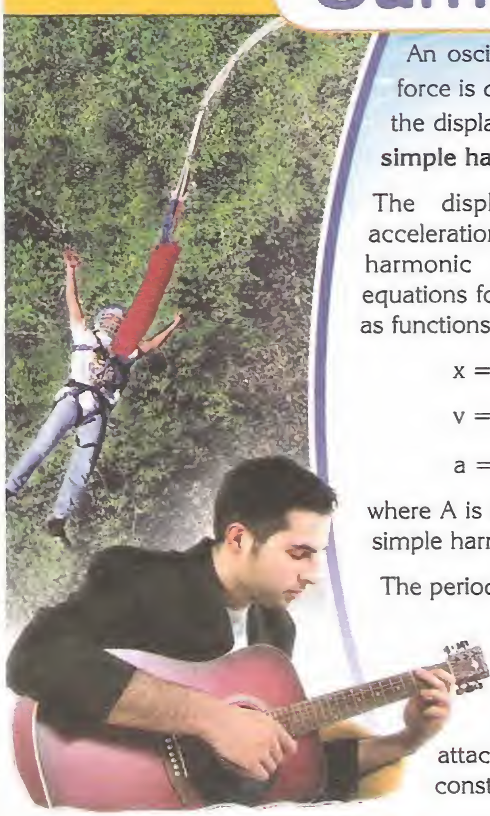
a) From the wave equation

$$\lambda = \frac{v}{f} = \frac{340 \text{ m}}{85000 \text{ Hz}} = 5\times10^{-3} \text{ m}$$

b) $$d = \frac{x}{2} = \frac{v\Delta t}{2} = \frac{(340 \text{ m/s})(0.01 \text{ s})}{2} = \frac{3.4 \text{ m}}{2} = 1.7 \text{ m}$$



# Summary



An oscillating system in which the restoring force is directly proportional to the negative of the displacement ($\vec{F}=-k\vec{x}$) is said to undergo **simple harmonic motion**

The displacement, the velocity and the acceleration of an object performing simple harmonic motion vary continuously. The equations for these quantities are thus expressed as functions of time as follows:

$$x = A\cos\omega t$$

$$v = -A\omega\sin\omega t$$

$$a = -A\omega^2\cos\omega t$$

where A is amplitude and $\omega$ is angular speed of simple harmonic motion.

The period of a mass-spring system is

$$T = 2\pi\sqrt{\frac{m}{k}}$$

where m is the mass of the object attached to the spring and k is the spring constant.

The period of a simple pendulum is

$$T = 2\pi\sqrt{\frac{L}{g}} = \lambda f$$

where L is the length of a simple pendulum and g is the gravitational acceleration.

A wave is a disturbance which transmits energy from one point to another by a vibrational mechanism. In general there are two types of **mechanical waves:**

**A transverse wave** is one in which the particles oscillate perpendicular to the direction of propagation of the wave

**A longitudinal wave** is one in which the particles oscillate parallel to the direction of propagation of the wave

**Wave velocity** is the distance travelled by a wave pattern in one period, T. This distance must be equal to the wavelength of the wave, $\lambda$. So its equation is

$$v = \frac{\lambda}{T}$$

where $\lambda$ is the wavelength, T is the period and v is the velocity of the wave.

# QUESTIONS AND PROBLEMS

9.1 Vibrations and Springs

9.2 Simple Harmonic Motion

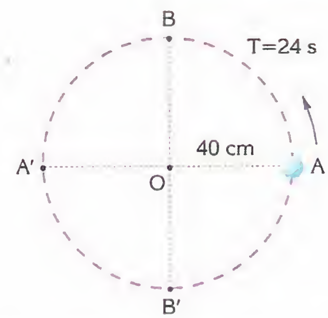
1. Decide whether the motions below are simple harmonic or not.
   a) The marching of soldiers from side to side on guard at the gate of a palace.
   b) A door which is regularly opening and closing on a windy day.
   c) A boat which rocks up and down on a rough sea.
   d) The motion of a piston in an automobile engine
   e) Rhythmic beats of a heart

2. Explain the similarity between simple harmonic motion and uniform circular motion.

3. Can a body that performs simple harmonic motion along a straight line have:
   a) velocity and displacement vectors in the same direction?
   b) acceleration and displacement vectors in the same direction?
   c) acceleration and velocity vectors in the same direction?

4. A mass-spring system performs 30 oscillations in 10 s. Find its
   a) period
   b) frequency.

5.

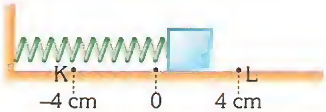



The period of an object of mass m, which performs uniform circular motion along a circular path of 40 cm radius, as shown in the figure, is 24 s.
   a) If the object starts to move from point A, how many seconds later will it pass point B?
   b) What are the projections of its velocity and acceleration on axis BB' while passing point A?
   c) What are the projections of its velocity and acceleration on axis BB' while passing point B?

6. The displacement equation for the motion of an object exhibiting simple harmonic motion is $x=0.6\cos(\pi t)$ m. Find the motion's
   a) amplitude,
   b) angular speed,
   c) frequency
   d) maximum speed
   e) maximum acceleration.

7. If an object performs simple harmonic motion with a frequency of $0.2\ s^{-1}$ and an amplitude of 10 cm, what is the maximum speed of the object?

8.




A mass-spring system is exhibiting simple harmonic motion between points K and L, as shown in the figure. If the mass has a value of 1 kg and covers the distance between points K and O in 3s, find

a) the period of the system

b) the maximum velocity of the system

c) the maximum acceleration of the system

d) the restoring force acting on the object at point K.

e) Assuming that the object starts moving at point L, state the equations of displacement, velocity and acceleration of the object as a function of time

f) Assuming that the object starts moving at point O towards point K, state the equations of displacement, velocity and acceleration of the object as a function of time

9. An object is exhibiting simple harmonic motion along the y-axis with a period of 1s. Assuming it started its motion by being compressed 50 cm from its equilibrium position

a) how long will it take to arrive at the equilibrium position for the first time?

b) find its maximum speed.

c) find its maximum acceleration.

d) state the equations of its motion as a function of time.

10. When a vertical mass-spring system, at rest in its equilibrium position, is struck slightly by a hammer, it compresses the spring by 20 cm. The system then performs simple harmonic motion with a period of 2 seconds.

Write down the equations for the position, velocity and acceleration of the object as a function of time.

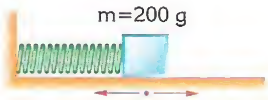
11. An object is exhibiting simple harmonic motion along the y-axis. The equation for its position as a function of time is $y=0.4 \sin(2.5\pi t)$ m. Find the position of the object along the y-axis at the following instants:

a) $t = 0$ s,

b) $t = 0.1$ s,

c) $t = 0.6$ s.

12. A 200 g object is attached to the end of a spring which rests on a smooth horizontal surface as shown in the figure. While in equilibrium, the object is pulled by 8 cm and released. If the maximum speed of the object exhibiting simple harmonic motion is 40 cm/s, find




a) the spring constant

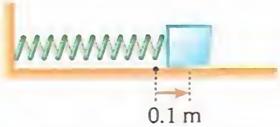
b) the maximum acceleration of the object

c) the speed of the object at the moment when it is 6 cm away from its equilibrium position.

13. An object performs simple harmonic motion between points K and L with a period of $4\pi$.

If the distance between points K and L is 10 cm, what is the speed of the object when it is 8 cm away from the equilibrium position?

**14.** A 2 kg object is attached to the end of a spring on a smooth plane, as shown in the figure, and is extended 0.1 m from its equilibrium position and then released. If the period of its simple harmonic motion is 2 s




a) what is the maximum force acting on the object?

b) what is the speed of the object at the moment its displacement is 0.05 m?

c) what is the force acting on the object at the moment when its displacement is 0.05 m?

**15.** When a mass-spring system is stretched 3 cm from its equilibrium position, it starts to perform simple harmonic motion with a frequency of 10 Hz. At what instants does it first pass the following positions?

a) $x_1$ = 1,5 cm,

b) $x_3$ = –3 cm,

**16.** A 1 kg object attached to a spring in the vertical is struck slightly by a hammer when it is at rest in its equilibrium position. This results in the mass gaining a velocity of 3 m/s. If the spring constant is 100 N/m, determine

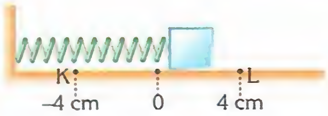
a) the period and frequency of the system

b) the amplitude of the system

c) the maximum acceleration of the system

d) State the equations for the position, velocity and acceleration as a function of time.

**17.**



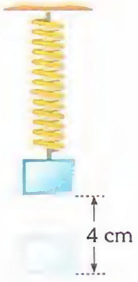

An object is exhibiting simple harmonic motion with an angular speed of 6 rad/s between points K and L, as shown in the figure. (Take $\pi$=3)

a) Assuming that the object starts moving at point L, draw the displacement-time, velocity-time and acceleration-time graphs of the system for a complete oscillation

b) Assuming that the object starts moving at point O towards point K, draw the displacement-time, the velocity-time and the acceleration-time graphs for a complete oscillation of the system.

## 9.3 Simple Harmonic Motion of a Mass-Spring System

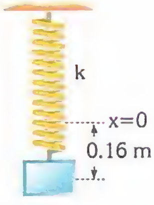
**18.** When a mass attached to a suspended spring is extended 4 cm from its equilibrium position, as shown in the figure, and then released, the system performs simple harmonic motion. If the spring reaches its lowest point 24 times in 96 s, find the system's




a) period

b) frequency.

**19.** When a 1 kg mass is suspended from a spring which is fixed to a ceiling, the spring extends by 0.1 m and reaches an equilibrium, as shown in the figure. Find:




a) the spring constant of the spring.

b) the period of the system if the mass is slightly extended from its equilibrium position and then released?

**20.**

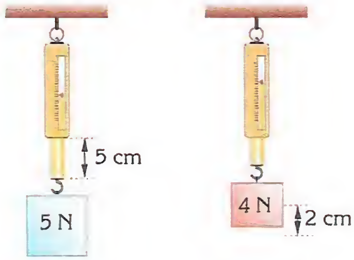



Figure - I Figure - II

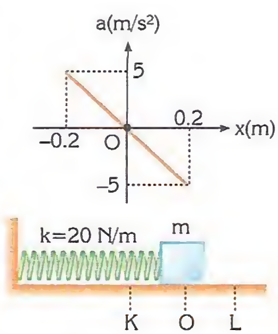
When a 5 N object is attached to a lab dynamometer, its spring extends by 5 cm and the system reaches an equilibrium, as shown in Figure I. A 4 N object is attached to the same dynamometer, extended by 2 cm from its equilibrium position, then released, as shown in Figure II. If the object oscillates vertically, find

a) the spring constant of the spring in the dynamometer

b) the frequency of the motion

c) the maximum velocity and the maximum acceleration of the system.

**21.** An object on a smooth plane, as shown in the figure, is exhibiting simple harmonic motion between points K and L, with an amplitude of 0.2 m. The spring constant is 20 N/m. The acceleration-displacement graph of the motion is also shown in the figure. Find

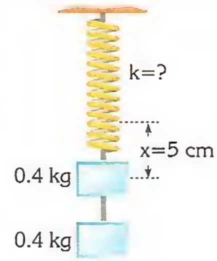



a) the period and frequency of the system

b) the mass of the object

c) the force acting on the object when it is at point K.

**22.** When two objects of mass 0.4 kg each are attached to the end of a spring whose spring constant is unknown, the spring extends by an amount of 5 cm, as shown in the figure.




a) Describe the motion of the object when the cord between the objects is cut whilst in equilibrium.

b) Find the spring constant and the period of the motion.

**23.**

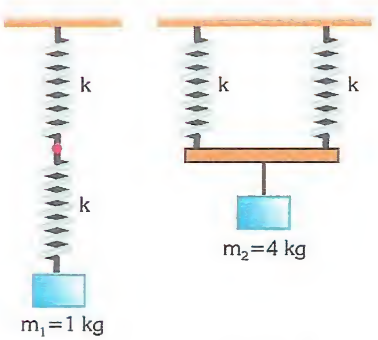
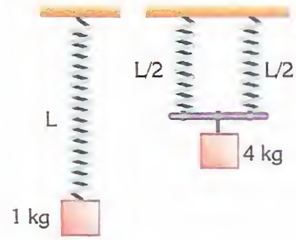



Figure - I Figure - II

Two spring-mass systems are constructed, as shown in Figure I and II, by the use of two identical springs of spring constant k. If the masses are $m_1=1$ kg and $m_2=4$ kg, what is the ratio of their periods, $T_1/T_2$?

24. A 1 kg object is suspended from a spring of length L. The period of its simple harmonic motion is measured to be 3 s. Later the spring is divided into two halves and the pieces are attached in parallel to a ceiling and a 4 kg object is suspended from the springs, as shown in the figure. When this object starts exhibiting simple harmonic motion, what will the period of its motion be? (The weight of the rod is neglected.)




25.

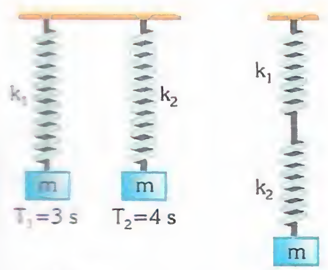
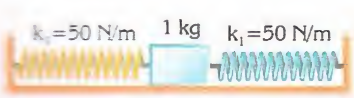



Figure - I Figure - II

The periods of the mass-spring systems in Figure-I of spring constants $k_1$ and $k_2$ are 3 s and 4 s, respectively. If the springs were connected to each other, as shown in Figure-II, what would the period of the system be?

26.




An object with mass m=1 kg is placed between two springs of spring constants $k_1=50$ N/m and $k_2=50$ N/m, as shown in the figure. Neglecting the effects of friction, calculate the period of the simple harmonic motion that will occur when the mass is pulled along the direction of the springs and then released. (Take $\pi=3$)

## 9.4 Simple Pendulum

27. A pendulum performs 40 oscillations in 50 s. What is its

a) period?

b) frequency?

28. How long must a simple pendulum be if it is to complete one oscillation in 2 seconds on the Earth?

29. What is the period of a 40 cm long simple pendulum

a) on the Earth where $g = 10$ m/s$^2$?

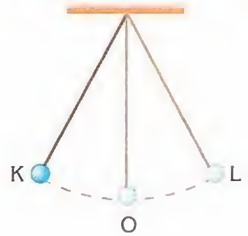
b) on the Moon where $g = 1,6$ m/s$^2$?

30. The period of a 0.5 m long simple pendulum is 2 s. What must the length of the pendulum be, so that its period is 1 s in the same plane?

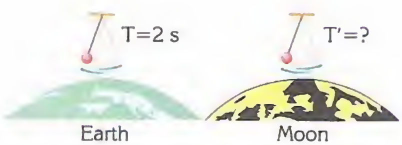
**31.** The simple pendulum in the figure is exhibiting simple harmonic motion between points K and L with a period T=2,8 s, as shown in the figure. If it starts oscillating from point K, at which point will the pendulum be after

a) 1,2 s?

b) 2,1 s?

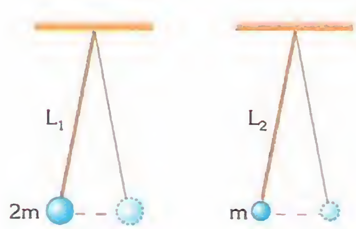



**32.**




If the period of a simple pendulum on the Earth is 2 s, what is its period on the Moon? (The gravitational acceleration on the Moon is 1/6 of that on Earth.)

**33.**

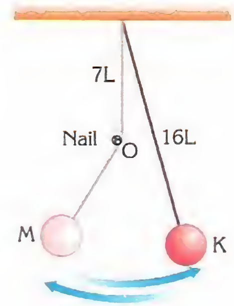



The simple pendula of lengths $L_1$ and $L_2$, as shown in the figure, are exhibiting simple harmonic motion. The ratio of their frequencies is $\frac{f_1}{f_2} = \frac{1}{4}$.

What is the ratio of the lengths of the pendula $\frac{L_1}{L_2}$?

**34.** A simple pendulum of length 16L is released from point K and can reach point M after being obstructed by a nail at point O, as shown in the figure. In this case the period of the simple pendulum is 7 s. What would the period of the pendulum be if the nail were removed?

(Take π to be 3)




**35.** The simple pendulum, which is suspended from the ceiling of a lift, oscillates with a period T, as shown in the figure. What is the period of the simple pendulum in terms of T while the lift is

a) ascending with an acceleration which equals g?

b) descending with an acceleration which equals g/2?

c) falling freely?

### 9.5 Mechanical Waves

**36.** Waves travel along a stretched string with a speed of 10 m/s. The end of the string is flicked up and down once every second.

What is the wavelength of the waves propagating along the string?

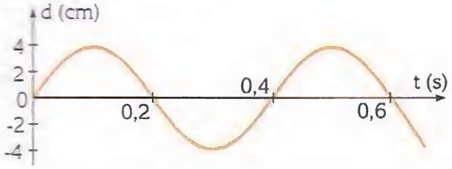
37. A periodic wave has a wavelength of 40 cm and a speed of 20 m/s.

What is the wave frequency?

38. A musical tone, produced on a piano, has a frequency of 400 Hz and a wavelength of 80 cm in air.

What is the wave speed?

39. The graph below represents the displacement of a particle in a wave with time.




If the wave is propagating with a speed of 50 cm/s, find its

a) amplitude

b) wavelength

c) period

d) frequency

40. A stone is dropped into a well. The sound of the splash is heard 3 s after releasing it.

What is the depth of the well if the speed of sound in air is 330 m/s and $g=10$ m/s$^2$?

41. A sonar impulse is emitted vertically downward from a stationary boat and the signal reflected from the sea bed is received on the boat after 10 seconds.

If the speed of sound in water is 1500 m/s, what is the depth of the water in this part of the sea?

42. Bats can detect small objects, such as insects, whose size is approximately equal to one wavelength of one of the frequencies that the bats emit.

If bats emit a sound wave at a frequency of 80 kHz, and the speed of sound waves in air is 340 m/s, what is the smallest size of insect they can detect?

43. A man in his boat moving up and down periodically, due to waves on the surface of the sea, measures 3 s time intervals as the boat travels from its highest point to its lowest point, a total distance of 1 m. The man observes that the distance between two successive crests is 6 m.

What is the speed of the waves?

44. The speed of sound in water at 20 °C is 1470 m/s and at 35 °C it is 1560 m/s. Suppose that a sound wave of frequency 450 Hz passes through a layer of water at 35 °C into a layer of water at 20 °C.

What will the change in wavelength be?
(Assume that the frequency remains the same.)

# ANSWERS KEY

## Chapter 1

### Units and Physical Quantities

1. $kg\frac{m^2}{s^2}$, Joules
2. $kg\frac{m^2}{s^3}$, watt
3. $\frac{m^3}{kg \cdot s^3}$
4. $\frac{kg}{s^2}$
5. $7.2\times10^5$ m/s
7. m=n=1
8. 0.675 kg
9. $10^{-17} \times 10^{-5}$ pico...
10. a) 3 s.f., b) 5 s.f.
    c) 2 s.f., d) 2 s.f.
11. $0.1\cdot10^{-6}$ mm to $0.5\cdot10^{-6}$ mm
14. about 53 min
15. a) 3 m/s; b) 8.9 kg; c) 6.02 Gm; d) 18 mJ; e) 5 μm
18. 2 units, 18 units
19. 500 N, 53° north of west
20. 16 m/s upwards
22. 10 units, 37° to the horizontal
25. ↘
    $R_x$=+2 units; $R_y$=–1 units
26. $\xrightarrow{\vec{P}}$
28. ←
    43.3 units in –y direction
29. 2 units $\xrightarrow{\vec{R}}$
30. 1 unit ←
33. 21.7 m/s; 12.5 m/s
34. 21.2 km; 21.2 km; 49.8 km; 41.8 km
35. 34.2 m due east, 94.0 m due south
36. a) 9.4 units, –58°; b) 8.5 units, 111° from +x axis
38. 10 units in +x direction
39. $5\sqrt{2}$ units

## Chapter 2

### Motion in One-Dimension

6. b) $\Delta x = +12$ m
7. a) 4.5 m; b) 3.0 m
8. 13 m
9. a) 20 m/s; b) 0.27 m/s; c) 54 km/h
11. 6 m/s
12. a) positive; b) positive; c) negative; d) at rest
16. b) 240 km
17. a) $9.5\times10^{12}$ km; b) 4.2 light year; c) 18068 years
18. 110 m
20. $2.0\times10^3$ s
21. 80 km/h
22. 30 s
23. 1.5 min; 3.5 km
24. 18 km
25. 32 s
26. x=3 + 2t
27. a) $v_1$=20 m/s, $v_2$=–30 m/s;
    b) 600 m, 1400 m;
    c) 920 m, 16 s
32. a) Both 3.1 m/s;
    b) 3.0 m/s, 2.0 m/s; c) 5.0 m/s, 0
34. 15 m/s or 54 km/h
35. a) 10.1 m/s; b) 6.4 s
37. a) 0.4 m/s; b) 0.4 m/s; c) zero
38. a) 40 m/s; b) 15 m/s; c) 30 m/s; d) 45 m/s; e) zero
39. a) 40 m/s; b) –60 m/s;
    c) zero; d) 60 m/s
40. a) 28 km/h; b) 20 km/h
41. a) 40 km/h
43. a) 4.8 m/s; b) zero
47. 5 m/s$^2$
48. –4 m/s$^2$
49. 1.5 m/s$^2$
50. a) –0.4 m/s$^2$; b) 4 m/s; c) 50 s
51. 15 m/s; 75 m
52. a) 50 s; b) 375 m
53. a) 2.4 m/s, 14.4 m; b) 75 s
54. a) 40 m/s, 325 m;
    b) 10 m/s, 175 m
55. 36 m
56. 17.5 m
57. a) 6.25 m/s$^2$; b) 4 s
59. 45 m
63. 7 m/s
65. a) 24 m/s; b) 24 m/s; c) 11.2 m/s
66. 1.35 m/s; $4.57\times10^5$ m/s$^2$
67. $3.2\times10^4$ km/h$^2$
68. 8 m/s; 0.4 m/s$^2$
69. a) three times; b) $\sqrt{3}$ times
70. 16.7 m/s$^2$
71. 2.5 km
72. a) 85; b) 32 m;
    c) 32 m/s, 24 m/s
73. 10 m/s
74. 150 m
75. $\frac{1}{9}$ m/s$^2$
79. 155 m/s or 558 km/h
80. a) 4 s; b) 40 m/s;
    c) 20 m/s, 20 m
81. a) 45 m/s; b) 9 s; c) 56.3 m
82. 120 m
83. a) 6 s; b) 40 m/s downwards;
    c) 75 m
84. 140 m
85. 0.18 s
86. a) 240 m; b) –70 m/s
87. 80 m
88. 12
89. 1.5 s
90. 480 m

## Chapter 3

### Motion in Two Dimensions

2. a) 40 m/s, 30 m/s; b) 3 s; c) 45 m; d) 240 m; e) 40 m, 41.2 m/s, 14°; f) 40 m/s, 41.2 m/s, –14°
5. 8.9 m
6. a) 18.4 s; b) 367 m
7. 12.1 m
10. a) 3 s; b) 60 m; c) 36 m/s, –56.3°; d) 45°

11. a) 0.4 s; b) 3 m/s; c) 5 m/s, –53°
12. a) 55 m/s; b) $1.1\times10^3$ m/s
13. $9\times10^6$ m
14. a) 2 s; b) 13 m; c) 12.7 m/s, –18.4° with, –y axis.
15. a) 20 m/s, b) 41.2 m/s, –67°; c) 1.2 s; d) 7.2 m; e) 80 m, –65 m
16. a) 5 m/s; b) 4 s; c) 92 m; d) 43,2 m/s, –85°
18. No, she cannot. ($h_{max}$=2.5 m)
19. a) 6 s; b) 33.5 m/s, 63,4°
20. 10 m
21. 26°
22. a) 21.4 m/s; b) 15.1 m
23. a) 12 m/s; b) 24 m/s; c) 20.8 m/s; d) 2.1 s
24. a) 20 s; b) 5000 m; c) 1875 m; d) freefall from rest
25. 0.18 m
26. a) 80 m; b) 6 s; c) 120 m
27. a) projectile motion; b) 4 s; c) 5 m/s
28. a) 0.9 s; b) 4.05 m; c) 2.52 m
29. a) 6 s; b) 180 m; c) 60 m
30. 30 m/s
32. a) 20 m/s; b) 20 m/s
33. a) 50 m/s; b) 50 m/s; c) 50 m/s
34. a) 7 m/s, 4 m/s; b) –1 m/s, –4 m/s; c) 5 m/s, 37° with vertical, 4 m/s
37. a) 2 m/s; b) 3 m/s
38. 102 km/h, 11.3 east of north
39. 5.03 m/s, 84.3° to the direction of passenger
40. 130 km/h, 22.6° east of north
41. 5[illegible], 37° north of east
42. a) 5.8 m/s, 31° to the vertical; b) 48 s, 144 m; c) 60 s
43. a) 50 m/s; b) 30 m/s
44. 600 m
45. 540 m
46. 18 s

## Chapter 4

## The Laws of Motion

3. a) 0, b) $4\sqrt{2}$ N ↑, c) 4 N ←
4. 5 N ←
18. 3 $m/s^2$ →
19. 5 $m/s^2$ →
20. a) 0; b) $2\sqrt{2}$ $m/s^2$ ↑; 2 $m/s^2$ →
21. 10 $m/s^2$ ↑
22. a) 10 $m/s^2$ →; b) 20.6 $m/s^2$ ↑
23. 6 N
24. 4000 N
25. 1000 $m/s^2$
26. 25 N
27. 18 m
28. 1000 N
29. 2080 N, 80 kg
30. a) 2 $m/s^2$; b) 12 N
31. a) 3 $m/s^2$; b) 9 N
32. a) F=200 N, T=100 N; b) F=220 N, T=110 N
33. a) 2.5 $m/s^2$; b) 7.5 N
34. 6 $m/s^2$
35. a) 2 $m/s^2$; b) 40 N
36. 50 N
37. a) 2 $m/s^2$; b) 24 N
38. a) 5 $m/s^2$; b) 30 N
39. 2.5 $m/s^2$
40. 2
41. a) 5 $m/s^2$; b) 4 N
42. a) 2 $m/s^2$; b) 16 N
43. a) 1 s; b) 2.5 m/s; c) 2.5 s
44. 312 N
45. a) $a_1$=12 $m/s^2$, $a_2$= 6 $m/s^2$; b) $T_1$= 12 N, $T_2$= 24 N
46. 20 $m/s^2$
47. 0.6
48. 2.5 $m/s^2$
49. 1.5 $m/s^2$
50. 0.5
51. 3/80
52. 25 m
54. a) 4 $m/s^2$; b) 12 N
55. 1.4 $m/s^2$
56. 7 $m/s^2$
57. a) 5 $m/s^2$; b) 15 N
58. 0.75
59. 1/3
60. a) 2 $m/s^2$; b) $T_1$=6 N, $T_2$=32 N
61. a) 5 $m/s^2$; b) 22.5 m
62. a) 0.8 $m/s^2$; b) 48 N
63. a) 20 $m/s^2$; b) 160 N
64. 37.5 N
66. a) 6 kg; b) 5 $m/s^2$
67. a) 400 N; b) 400 N; c) 400 N; d) 480 N; e) 320 N; f) 0
68. 50 N
69. a) 4 $m/s^2$ ↑; b) 4.5 $m/s^2$ →
70. $a_1$=4.4 $m/s^2$, upward; $a_2$=0.4 $m/s^2$, downward
71. 6/5
72. a) 9 $m/s^2$; b) 800 N

## Chapter 5

## Torque and Equilibrium

2. 3 kg
3. 4 kg
4. a) 100 N; b) 60 N
5. 40 N
6. 0.5
7. $T_1$=44 N, $T_2$=40 N, $T_3$=50 N, θ=27°
8. T=$30\sqrt{2}$, $R_1$=60 N, $R_2$=30 N, $R_3$=30 N
11. 10 Nm
12. a) 0; b) 2 Fd
13. 72 Nm
14. 16 Nm
15. a) $\tau_1$ = 20 Nm, $\tau_2$ = 90 Nm, $\tau_3$ = –70 Nm; b) 40 Nm
16. 6 Nm
17. –6 Nm
18. –45 Nm
20. 100 N
21. 40 N
22. a) 250 N; b) $R_x$=250 N, $R_y$=500 N
23. $F_A$=10 N, $F_B$=30 N
24. 40 N, 20 N
25. a) 25 N; b) $F_x$=15 N; $F_y$=10 N
26. $\frac{T_1}{T_2}=\frac{1}{2}$
27. a) 30 cm; b) 30 N
28. a) 5 units from P; b) 6 P
29. 35 N, 21 N
30. $\frac{25}{8}$
31. 56 N, 16.4 N, 44.8 N

32. 500 N
33. 15 N, 34 N
34. $\frac{3L}{4}$
38. $x_{CM}=3$ cm, $y_{CM}=4$ cm
39. $x_{CM}=1$ cm, $y_{CM}=1$ cm
40. $x_{CM}=\frac{11}{6}$ m, $y_{CM}=\frac{9}{6}$ m
41. $x_{CM}=2.3$ cm, $y_{CM}=1$ cm
42. 3.1 cm
43. 0.25 cm
44. $x_{CM}=1.54$ cm, $y_{CM}=1.72$ cm
45. $\frac{1}{2}r$
46. between points M and N
47. 3762 km
48. $x_{CM}=\frac{125}{18}$, $y_{CM}=\frac{185}{18}$
49. $\frac{r}{5}$
50. $\frac{3r}{14}$

## Chapter 6

## Work and Energy

4. 100 J
5. 16 J
6. $W_1=W_2=W_3$
7. a) 200 J; b) –80 J; c) 120 J
8. a) $W_1=160$ J, $W_2=0$, $W_3=-120$ J; b) 40 J
9. a) 0; b) 200 J
10. a) 75 J; b) –60 J; c) No
11. a) 200 J; b) 0; c) –200 J
12. a) 12 J; b) 6 J; c) –24 J; d) –6 J
13. 75 J
14. a) 1 000 J; b) 10 000 000 J
17. a) 8 J; b) 50 J
19. $4\times10^2$ J
20. 2 m/s
21. a) 250 J; b) 5 m/s; c)10 s
22. a) 144 J; b) 8 m/s, c) $4\sqrt{3}$ m/s
23. a) –10 J; b) 6 m/s
24. a) 20 N; b) 1660 J; c) 1600 J; d) 40 m/s
25. –25 J
26. a) 44 J; b) 44 J
27. a) –125 J; b) 200 J; c) 75 J; d) 5.48 m/s
28. 10 m
29. a) –150 J; b) 0.75
30. a) –200 J; b) 5 m/s
31. a) –36 J; b) 28 J
32. a) –40 J; b) 30 J; c) 50 J
33. at L –200 J, at M 400 J
34. a) 60 J; b) –60 J
35. a) 120 J; b) –120 J
36. a) 600 J; b) 600 J
37. 4 J
38. a) $PE_A=3$ J, $PE_B=10$ J; b) 20 J
39. a) $m_1=0.1$ kg; b) $m_2=0.5$ kg
40. 10 m/s
41. 5 m
42. a) 40 J; b) 10 m/s
43. 40 m
44. 30 m/s
45. $2\sqrt{2}$
46. 25 N
47. a) 10 m/s; b) 40 J
48. a) 4 J; b) 16 J
49. $PE_K=80$ J, $PE_L=40$ J
50. a) $10^5$ N/m; b) 6.4 J; c) 3.2 J
51. 0.2 m
52. a) 0.5 m; b) $\frac{\sqrt{3}}{3}$ m
53. 0.3 m
54. 0.2 m
55. a) 500 N/m; b) 90 J
56. 0.3 m
57. a) 2 m/s; b) 0.2 m
58. 6 m/s
59. a) 60 J; b) 100 J
60. a) 10 m/s; b) 40 m
61. 3.1 m
62. a) 5 m/s; b) 0.4 m
63. a) 6 m/s; b) 60 cm
64. a) 3200 J; b) 160 W
65. 300 W
66. $P_x=100$ W, $P_y=600$ W, $P_z=150$ W
67. a) 3750 kJ; b) 500 hp
68. 500 W
69. a) $3\times10^5$ J; b) 40 hp
70. 200 W, 90 %
71. $5\times10^4$ W

## Chapter 7

## Momentum and Impulse

2. 4.5 kgm/s
3. $18.2\times10^{-26}$ kgm/s
4. 20 m/s
5. a) 0; b) $2\sqrt{2}$ kgm/s
6. 48 000 kgm/s
8. 5 Ns
9. 4 m/s
10. 1 800 N
11. 1000 Ns
12. a) $1.6\times10^4$ Ns; b) $1.2\times10^3$ Ns; c) $1.6\times10^3$ N
13. a) $5\sqrt{2}$ kgm/s; b) $25\sqrt{2}$ N
14. $10\sqrt{2}$ Ns
15. 10 000 N
16. 20 kgm/s
17. a) 10 Ns; b) 100 N
18. 10 kgm/s
19. 50 kgm/s
20. 5 m/s due east
21. a) 6 Ns; b) 0.2 m/s
22. 3 m/s due east
23. 16 m/s to the left
24. a) 1.6 m/s to the left; b) 24 N
25. 11 m/s to the right
26. a) 4 m/s; b) 0.25 m/s
27. 2.5 m/s to the right; b)
28. 8 m/s
29. 3 m/s
30. a) 0; b) 2 m/s
31. 0.5 m/s to the left
32. 2.4 m/s
33. 2 m/s in direction 2
34. $\frac{2}{3}$
35. 200 m/s
36. a) 10 m/s; b) 5 m; c) 5 Ns
37. 5 m/s to the left
38. a) 1 m/s; b) 10 cm
39. 20 cm
40. $u_1=3$ m/s to the left, $u_2=2$ m/s to the right
41. $u_1=2$ m/s to the left, $u_2=4$ m/s to the right
42. $u_1=13$ m/s to the left, $u_2=2$ m/s to the right

43. $u_1=5$ m/s to the right, $u_2=8$ m/s to the right
44. $u_1=17$ m/s to the left, $u_2=7$ m/s to the left
45. 15 kg
46. 4 m/s to the right
47. $v_c= 2$m/s
48. a) 2.5 m/s; b) 18.75 J
49. 2 m/s due east
50. a) $u_1=0.5$ m/s to the left; b) 4.2 m
51. 10 m/s and 37° due south-east
52. 10 m/s to the right
53. 3 m/s (in –y direction)
54. 0
55. 4 m/s in y direction
56. 2 N
57. 0.3 m/s slower
58. 20 m/s
59. 10 m/s in the positive direction
60. 5 m/s
61. 35 m/s
62. 4 m/s
63. a) 4.8 Ns; b) 0.1 m/s
64. 40 m/s
65. 600 m/s
66. 200 m/s

## Chapter 8
## Uniform Circular Motion and the Universal Law of Gravity

4. a) 6 s; b) 1/6 Hz; c) 1 rad/s; d) 0.5 m/s
5. a) 1/60 s; b) 60 Hz; c) 360 rad/s
6. $T_{hour\ hand}=43200$ s; $T_{min.\ hand}=3600$ s; $T_{second\ hand}=60$ s
7. a) 25 Hz, 1/25 s; b) 15 m/s; c) 45 m/s
8. a) 1/20 s; b) 300 m/s
9. 26 666.6 m/s
10. a) $\omega_1= \omega_2= \omega_3$; b) $v_3= 3/2v^2= 3v_1$
11. $T_2=12$ s
12. 3/2
13. 120°
15. a) 3 m/s; b) $3\sqrt{2}$ m/s; c) 6 m/s; d) 0
16. a) 3 m/s; b) 3/34 m/s$^2$; c) 1.5 m/s$^2$; e) 3 N
17. a) 4.5 m/s$^2$; b) 9 N
18. 72 N
19. a) 1.8 N; b) 0.36
20. 1
21. 2 m/s
22. 5/8 m
23. a) 5 rad/s; b) 3 N; c) 5 N
24. 2
25. 3 m/s
26. 4800 N, lighter
27. a) 13125 N; b) 1050 N
28. 2T
29. 2.5mg
30. 0.45
31. 20 m/s
32. b) 0.75; c) 9000 N
33. 53°
34. 10 m/s
35. 0.25
40. $2\times10^{18}$ N
41. $40\times10^{-4}$ N
42. $53.4\times10^{-9}$ N
43. 4
44. 26 m/s$^2$, 3.7 m/s$^2$
45. $54R_E$ from the Earth
46. 345 km
47. $26\times10^7$ m
48. a) 4000 kg; b) 1700 N
49. a) 86400 s; b) 3000 m/s
50. a) 29700 m/s; b) $1.97\times10^{30}$ kg
51. a) $7.3\times10^{-5}$ rad/s; b) 465.2 m/s, 232.6 m/s; c) 1.7 N
52. a) 8000 m/s; b) 83.7 min
53. 2/3 N
54. $-5\times10^{-14}$ J
55. a) $-13.4\times10^9$ J; b) $6.7\times10^9$ J; c) $-6.7\times10^9$ J; d) 3.7 km/s
56. a) $E_b= 6.3\times10^{11}$ J, $v_{esc}=1.12\times10^4$ m/s; b) $E_b= 1.8\times10^{15}$ J, $v_{esc}=6\times10^5$ m/s

## Chapter 9
## Simple Harmonic Motion and Mechanical Waves

4. a) 1/3 s; b) 3 Hz
5. a) 6 s; b) v=0.1 m/s, a=0; c) v=0, a=0.03 m/s$^2$
6. a) 0.6 m; b) 3.14 rad/s; c) 1/2 Hz; d) 1.9 m/s; e) 5.9 m/s$^2$
7. 0.13 m/s
8. a) 12 s; b) 0.02 m/s; c) 0.01 m/s$^2$; d) 0.01 N
9. a) 0.25 s; b) 3.14 m/s; c) 19.7 m/s$^2$
11. a) 0; b) $0.2\sqrt{2}$ m; c) –0.4 m
12. a) 5 N/m; b) 2 m/s$^2$; c) $0.1\sqrt{7}$ m/s
13. $3\times10^{-2}$ m/s
14. a) 1.97 N; b) 2.7 m/s; c) 0.98 N
15. a) 3/20 s; b) 1/20 s
16. a) 0.628 s, 1.6 Hz; b) 0.3 m; c) 30 m/s$^2$
18. a) 4 s; b) 1/4 Hz
19. a) 100 N/m; b) 0.628 s
20. a) 100 N/m; b) 2.5 Hz; c) 0.314 m/s, 4.9 m/s$^2$
21. a) 1.256 s, 0.8 Hz; b) 0.8 kg; c) 4 N
22. b) k=160 N/m, T=0.314 s
23. 1
24. $3\sqrt{2}$ s
25. 5.5 s
26. 0.628 s
27. a) 1.25 s; b) 0.8 Hz
28. 1 m
29. a) 1.256 s; b) 3.14 s
30. 0.125 m
31. a) between O and L; b) at point O
32. $2\sqrt{6}$ s
33. 2
34. 8 s
35. a) $T/\sqrt{2}$; b) $T\sqrt{2}$; c) infinity
36. 10 m
37. 50 Hz
38. 320 m/s
39. a) 4 cm; b) 20 cm; c) 0.4 s; d) 2.5 Hz
40. 41 m
41. 7500 m
42. $4.25\times10^{-3}$ m
43. 1 m/s
44. 0.2 m



## APPENDIX 1 Physical Constants and Data

| | | | |
|---|---|---|---|
| Speed of light | $c=2.997925\times10^{8}$ m/s | Mass of earth | $5.98\times10^{24}$ kg |
| Gravitational constant | $G=6.67\times10^{-11}$ N·M$^2$/kg$^2$ | Average density of earth | 5.570 kg/m$^3$ |
| Electron charge | $e=1.60218\times10^{-19}$ C | Average earth-moon distance | $3.84\times10^{8}$ m |
| Permeability constant | $\mu_0=4\pi\times10^{-7}$ N/A$^2$ | Average earth-sun distance | $1.496\times10^{11}$ m |
| Standard gravitational acceleration | $g=9.80665$ m/s$^2$$=32.17$ ft/s$^2$ | | |

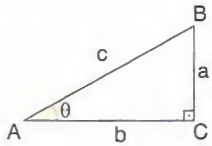
## APPENDIX 2 Trigonometric Table

| θ (Deg.) | θ (Rad.) | sin θ | cos θ | tan θ | θ (Deg.) | θ (Rad.) | sin θ | cos θ | tan θ |
|---|---|---|---|---|---|---|---|---|---|
| 0° | 0.000 | 0.000 | 1.000 | 0.000 | | | | | |
| 1° | 0.017 | 0.017 | 1.000 | 0.017 | 46° | 0.803 | 0.719 | 0.695 | 1.036 |
| 2° | 0.035 | 0.035 | 0.999 | 0.035 | 47° | 0.820 | 0.731 | 0.682 | 1.072 |
| 3° | 0.052 | 0.052 | 0.999 | 0.052 | 48° | 0.838 | 0.743 | 0.669 | 1.111 |
| 4° | 0.070 | 0.070 | 0.998 | 0.070 | 49° | 0.855 | 0.755 | 0.656 | 1.15 |
| 5° | 0.087 | 0.087 | 0.996 | 0.087 | 50° | 0.873 | 0.766 | 0.643 | 1.192 |
| 6° | 0.105 | 0.105 | 0.995 | 0.105 | 51° | 0.890 | 0.777 | 0.629 | 1.235 |
| 7° | 0.122 | 0.122 | 0.993 | 0.123 | 52° | 0.908 | 0.788 | 0.616 | 1.28 |
| 8° | 0.140 | 0.139 | 0.990 | 0.141 | 53° | 0.925 | 0.799 | 0.602 | 1.327 |
| 9° | 0.157 | 0.156 | 0.988 | 0.158 | 54° | 0.942 | 0.809 | 0.588 | 1.376 |
| 10° | 0.175 | 0.174 | 0.985 | 0.176 | 55° | 0.960 | 0.819 | 0.574 | 1.428 |
| 11° | 0.192 | 0.191 | 0.982 | 0.194 | 56° | 0.977 | 0.829 | 0.559 | 1.483 |
| 12° | 0.209 | 0.208 | 0.978 | 0.213 | 57° | 0.995 | 0.839 | 0.545 | 1.540 |
| 13° | 0.227 | 0.225 | 0.974 | 0.231 | 58° | 1.012 | 0.848 | 0.530 | 1.600 |
| 14° | 0.244 | 0.242 | 0.970 | 0.249 | 59° | 1.030 | 0.857 | 0.515 | 1.664 |
| 15° | 0.262 | 0.259 | 0.966 | 0.268 | 60° | 1.047 | 0.866 | 0.500 | 1.732 |
| 16° | 0.279 | 0.276 | 0.961 | 0.287 | 61° | 1.065 | 0.875 | 0.485 | 1.804 |
| 17° | 0.297 | 0.292 | 0.956 | 0.306 | 62° | 1.082 | 0.883 | 0.469 | 1.881 |
| 18° | 0.314 | 0.309 | 0.951 | 0.325 | 63° | 1.100 | 0.891 | 0.454 | 1.963 |
| 19° | 0.332 | 0.326 | 0.946 | 0.344 | 64° | 1.117 | 0.899 | 0.438 | 2.050 |
| 20° | 0.349 | 0.342 | 0.940 | 0.364 | 65° | 1.134 | 0.906 | 0.423 | 2.145 |
| 21° | 0.367 | 0.358 | 0.934 | 0.384 | 66° | 1.152 | 0.914 | 0.407 | 2.246 |
| 22° | 0.384 | 0.375 | 0.927 | 0.404 | 67° | 1.169 | 0.921 | 0.391 | 2.356 |
| 23° | 0.401 | 0.391 | 0.921 | 0.424 | 68° | 1.187 | 0.927 | 0.375 | 2.475 |
| 24° | 0.419 | 0.407 | 0.914 | 0.445 | 69° | 1.204 | 0.934 | 0.358 | 2.605 |
| 25° | 0.436 | 0.423 | 0.906 | 0.466 | 70° | 1.222 | 0.940 | 0.342 | 2.747 |
| 26° | 0.454 | 0.438 | 0.899 | 0.488 | 71° | 1.239 | 0.946 | 0.326 | 2.904 |
| 27° | 0.471 | 0.454 | 0.891 | 0.510 | 72° | 1.257 | 0.951 | 0.309 | 3.078 |
| 28° | 0.489 | 0.469 | 0.883 | 0.532 | 73° | 1.274 | 0.956 | 0.292 | 3.271 |
| 29° | 0.506 | 0.485 | 0.875 | 0.554 | 74° | 1.292 | 0.961 | 0.276 | 3.487 |
| 30° | 0.524 | 0.500 | 0.866 | 0.577 | 75° | 1.309 | 0.966 | 0.259 | 3.732 |
| 31° | 0.541 | 0.515 | 0.857 | 0.601 | 76° | 1.326 | 0.970 | 0.242 | 4.011 |
| 32° | 0.559 | 0.530 | 0.848 | 0.625 | 77° | 1.344 | 0.974 | 0.225 | 4.331 |
| 33° | 0.576 | 0.545 | 0.839 | 0.649 | 78° | 1.361 | 0.978 | 0.208 | 4.705 |
| 34° | 0.593 | 0.559 | 0.829 | 0.675 | 79° | 1.379 | 0.982 | 0.191 | 5.145 |
| 35° | 0.611 | 0.574 | 0.819 | 0.700 | 80° | 1.396 | 0.985 | 0.174 | 5.671 |
| 36° | 0.628 | 0.588 | 0.809 | 0.727 | 81° | 1.414 | 0.988 | 0.156 | 6.314 |
| 37° | 0.646 | 0.602 | 0.799 | 0.754 | 82° | 1.431 | 0.990 | 0.139 | 7.115 |
| 38° | 0.663 | 0.616 | 0.788 | 0.781 | 83° | 1.449 | 0.993 | 0.122 | 8.144 |
| 39° | 0.681 | 0.629 | 0.777 | 0.810 | 84° | 1.466 | 0.995 | 0.105 | 9.514 |
| 40° | 0.698 | 0.643 | 0.766 | 0.839 | 85° | 1.484 | 0.996 | 0.087 | 11.43 |
| 41° | 0.716 | 0.656 | 0.755 | 0.869 | 86° | 1.501 | 0.998 | 0.070 | 14.301 |
| 42° | 0.733 | 0.669 | 0.743 | 0.900 | 87° | 1.518 | 0.999 | 0.052 | 19.081 |
| 43° | 0.750 | 0.682 | 0.731 | 0.933 | 88° | 1.536 | 0.999 | 0.035 | 28.636 |
| 44° | 0.768 | 0.695 | 0.719 | 0.966 | 89° | 1.553 | 1.000 | 0.017 | 57.290 |
| 45° | 0.785 | 0.707 | 0.707 | 1.000 | 90° | 1.571 | 1.000 | 0.000 | ∞ |

## APPENDIX 3 Mathematics for Physics

### Trigonometric Relations in a Right Triangle

In a right triangle ABC, where the sides a and b are perpendicular each other, c is hypotenuse and θ is an acute angle;




a) Sine of the angle θ is equal to the ratio of the opposite side of this angle (side a) to the hypotenuse (side c).

$$\sin\theta = \frac{a}{c}$$

b) Cosine of the angle θ is equal to the ratio of the adjacent side of this angle (side b) to the hypotenuse (side c).

$$\cos\theta = \frac{b}{c}$$

c) Tangent of the angle θ is equal to the ratio of the opposite side to the adjacent side of this angle.

$$\tan\theta = \frac{a}{b}$$

Tangent θ is also equal to the ratio of the sine of θ to the cosine of θ.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{b}$$

**Some Trigonometric Relations**

a) If $\alpha + \beta = 90°$ then, $\sin\alpha = \cos\beta$

b) If $\alpha + \beta = 180°$ then, $\sin\alpha = \sin\beta$

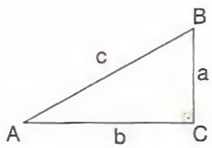
$\cos\alpha = -\cos\beta$

### The Theorems of Pythagorean, Sinus and Cosines

**Pythagorean theorem:**

In a right triangle ABC, where a and b are perpendicular sides and c is the hypotenuse, "the square of hypotenuse is equal to the sum of the squares of perpendicular sides". Hence the Pythagorean theorem can be expressed as follows:

$$c^2 = a^2 + b^2$$

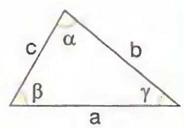



**Cosine and Sine Theorems :**

Cosine and sine theorems defined for any triangle ABC are as follows:

Cosine theorem :

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$




Sine theorem : $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$

# INDEX

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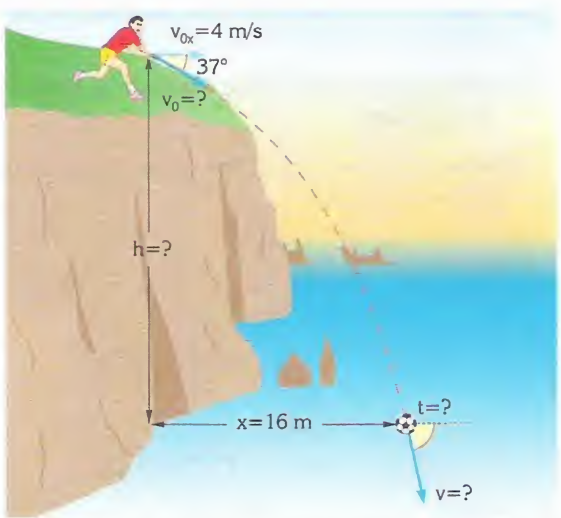
b) the skier's final velocity at the moment of landing on the ground.

c) the time the skier takes to reach his maximum height.

d) the skier's maximum height above the level of the ramp.

e) the skier's horizontal and vertical displacement from the time of leaving the ramp until he reaches the ground. (Take $g = 10\ m/s^2$ )

**16.**




A child standing at the edge of a cliff at the seaside hits a ball into the sea, 16 m away from the cliff, with a stone, as shown into the figure. The child throws the stone at an angle of –37° with the +x axis. If the horizontal component of the initial velocity of the stone is $v_{0x}$=4 m/s, find

a) the initial velocity of the stone

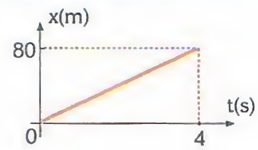
b) the time elapsed before the stone strikes the ball,

c) the height from which the stone was thrown

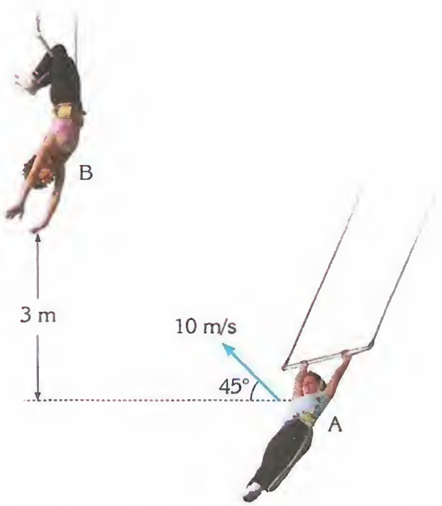
d) the velocity of the stone as it strikes the ball.

($g = 10\ m/s^2$)

**17.** The horizontal position-time graph of an object which is thrown upwards making an angle of 45° with the horizontal, is as shown in the figure. Draw vertical and horizontal velocity-time graphs of the object.

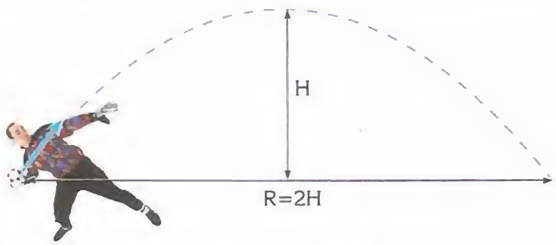



**18.**




Acrobat A releases the trapeze with an initial velocity of 10 m/s at an angle of 45° above the horizontal, as shown in the figure. Acrobat B, who hangs by her knees on another trapeze, is stationary 3.0 m above A's initial launch point. Can acrobat B catch acrobat A?

**19.**

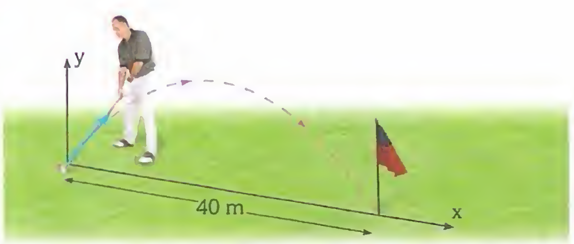



A ball is thrown at an initial velocity of v making an angle with the horizontal. At its maximum height H, the magnitude of its velocity is 15 m/s. The horizontal range R of the ball is twice the maximum height (that is R=2H).

a) For how long is the ball in flight?

b) What is the magnitude and direction of the ball's initial velocity v?

**20.**

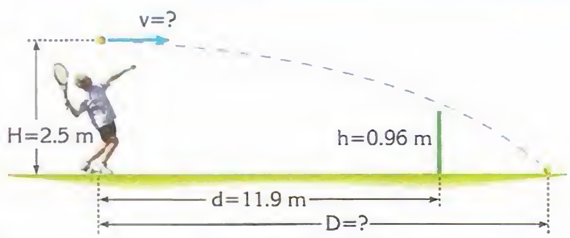



All objects in projectile motion achieve a maximum horizontal range when they are projected at 45° to the horizontal (if air resistance is ignored). A golf player strikes a golf ball so that it has a range of 40 m, as shown in the figure. Find the ball's maximum height.

**21.** Recep throws a ball at 8 m/s to Adile who is 5 m away. Adile catches the ball at the same height as Recep threw it. At what angle did he throw the ball?

(Hint: $\sin 2\alpha = 2 \sin\alpha \cos\alpha$)

**22.**




A tennis player strikes a ball with a racket during a serve. The ball was at H=2.5m above the ground and received an initial velocity v parallel to the ground's surface, as shown in the figure.

a) What must the minimum velocity of this ball be for it to pass over the net, which is h=0.96m high? The net is at d=11.9m from the serving position.

b) At this minimum velocity what will the distance D, from the service position to the point at which the ball strikes the ground, be?

**23.** An object is thrown horizontally from an altitude with an initial speed of 12 m/s. When it makes an angle of 60° with the horizontal

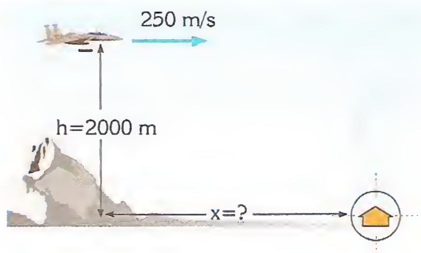
a) What is the x component of its velocity?

b) What is the velocity of the object?

c) What is the y component of its velocity?

d) How long does it take before it makes an angle of 60° with the horizontal?

**24.**




A military plane which flies at a velocity of 250 m/s, at a height of 2000 m, drops a bomb as it approaches a target, and strikes it, as shown in the figure.

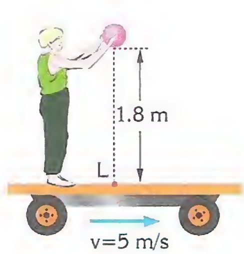
a) How many seconds after being dropped does the bomb strike the target?

b) How many metres away is the target from the vertical projection of the point where the bomb is released, x?

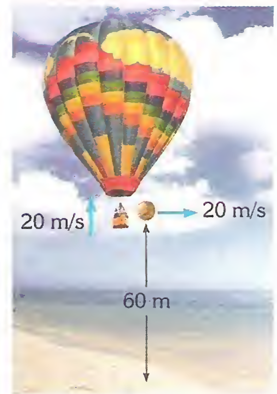
c) What is the height of the bomb 5 s after being dropped?

d) What type of motion does the bomb appear to have to the pilot?

**25.** A man is standing on a railway car which moves at a constant velocity of 5 m/s. He releases an object in his hand from a height of 1.8 m, as shown in the figure. At the same instant that the man releases the object, the railway car starts slowing down at a deceleration of 1 $m/s^2$. At what horizontal distance from point L on the car does the object strike the car? (Assume that the car is long enough)




**26.** An object is projected horizontally with a velocity of 20 m/s at a height of 60 m, from a hot air balloon, which rises at a constant velocity of 20 m/s.

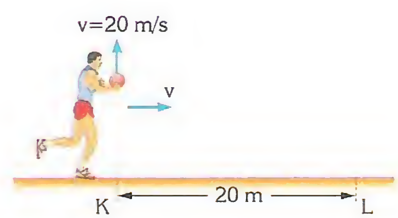



a) What maximum height does the object reach relative to the ground?

b) Find how long the object is in the air.

c) How many metres does the object cover in the horizontal direction before it strikes the ground?

**27.**



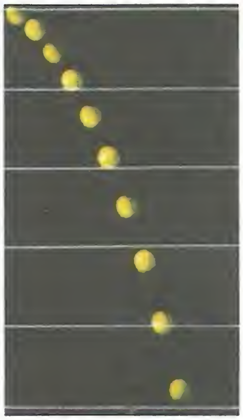

While a gymnast runs at a constant velocity v, he throws a ball in his hand vertically upwards at a velocity of 20 m/s, as shown in the figure. If the gymnast catches the ball when he arrives at point L

a) What type of motion does the ball exhibit relative to the Earth?

b) Find the time the object is in the air.

c) What is the velocity of the gymnast?

**28.** A ball is thrown horizontally and photographed using a multi-flash camera. Its motion is shown sequentially in the right-hand figure. The time interval between each snapshot is 0.1 s.

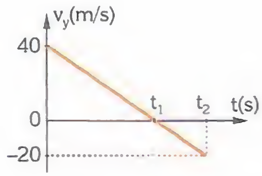


a) Find the time of travel of the ball.

b) Calculate the height from which the ball was thrown.

c) If the ball is thrown horizontally at 2.8 m/s, calculate the range of the projectile.

**29.**



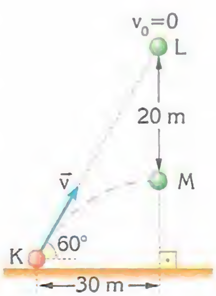

An object is thrown at an angle of 53° to the horizontal towards a hill. The graph shows how its vertical component of velocity changes in time.

a) Find the time the object is in the air.

b) What is the horizontal distance travelled by the stone?

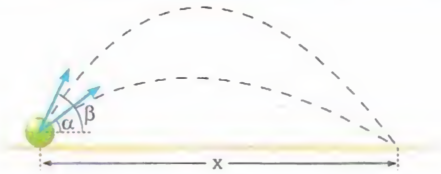
c) How many metres above the point where it was thrown does the stone land?

**30.** At the moment a red ball is thrown from point K with a velocity of v, a green ball is released from point L, as shown in the figure. The balls collide at point M. What is the initial velocity v of the red ball?




(Note that the red ball always strikes the green ball regardless of its initial velocity, provided that both are thrown and released at the same time. Can you explain why?).

**31.**




The horizontal range of an object in projectile motion is given as $x=v_{0x}\cdot t$ where $v_{0x}=v_0\cdot\cos\beta$ is the horizontal component of the initial velocity and β is the angle the object travels to the horizontal, as shown in the figure. The flight time of the same object can be also written as $t=\frac{2v_{0y}}{g}$ where $v_{0y}$ is the vertical component of the initial velocity. Show that the object covers the same horizontal range when it is thrown at a complementary angle α. (That is, $\alpha+\beta=90°$)

## 3.2 Relative Motion

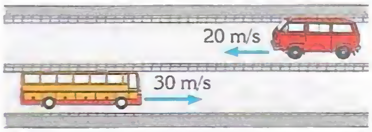
**32.**


A car and a motorcycle moving along a straight street in the same direction have velocities relative to the ground, as shown in the figure.

a) What is the velocity of the car relative to the motorcycle?

b) What is the velocity of the motorcycle relative to the car?

**33.**




While a bus moves at a velocity of 30 m/s due east, a van from the opposite direction moves at a velocity of 20 m/s, both are relative to the ground.

a) What is the velocity of the bus relative to the van?

b) What is the velocity of the van relative to the bus?

c) What is the velocity of the van relative to the bus after they pass each other?

**34.** A river flows at a velocity of 3 m/s. A fish has a velocity of 4 m/s in still water. (That is the velocity relative to the river.) What is the velocity of the fish relative to the Earth and the river

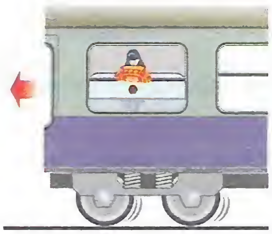
a) If it swims in the same direction as the river?

b) If it swims in the opposite direction to the river?

c) If it swims perpendicularly to the river flow?

**35.** A bus travels at a constant velocity. If a child in the bus jumps upwards from rest, where does she land relative to the point from which she jumped on the bus? If the bus was accelerating, what would your answer be?

**36.** A child drops a stone from the window of a fast-moving train.



a) What type of motion does the stone have relative to the child?

b) What sort of motion does the stone have relative to an observer on the ground?

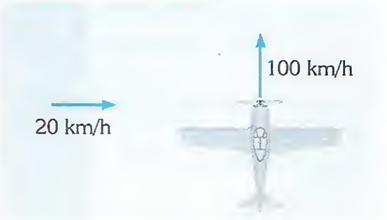
c) What sort of motion does the stone have relative to an observer in another train, travelling twice as fast as the first one and in the same direction?

**37.** Two canoe riders who are able to ride at equal velocities on still water, are sailing their canoes on a constantly flowing river. One is sailing in the same direction as the water flow and the other is sailing in the opposite direction to the flow. An observer on the river bank perceives the velocities of the canoes to be 5 m/s and 1 m/s.

a) Find the velocity of flow of the river.

b) Find the velocities of the canoes on still water.

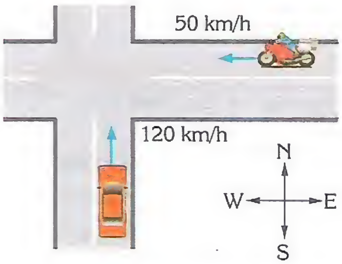
**38.**




The speedometer of an airplane flying due north reads 100 km/h. The wind blows due east at a constant velocity of 20 m/s relative to the Earth, as shown in the figure. What is the velocity of the aeroplane relative to the Earth?

**39.** A passenger in a train walks perpendicular to the direction of the train at a velocity of 0.5 m/s. If the train moves at a velocity of 5m/s relative to the ground, what is the velocity of the passenger relative to the ground?

**40.**




A motorcycle which travels due west at a velocity of 50 km/h and a car which travels due north at a speed of 120 km/h are approaching a crossroads, as shown in the figure. At what velocity does the motorcyclist observe the car moving at?

# MECHANICS

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